EQRC introduces the Implied Volatility Parametrization for Pricing, Trading & Risk Management for Vanilla Options Strategies

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Context

The area of vanilla options automated market making is considered amongst the most technically challenging markets in the world. Its mathematical complexity, and more specifically the challenges associated to the modelling of the Implied Volatility (IV) Surface (IVS) requires a dual expertise in mathematical modelling but also an understanding associated to the asynchronous nature of different strikes and tenors and their co-dependencies. To exacerbate these points volatility is substantially more important in some markets (eg: commodities, single stocks, cryptocurrency) and some points of IVS are also far less liquid and not normalized either. However, these technical challenges also means that they filter out the majority of market participants which therefore offers, as a result, a considerable financial opportunity if complexity is handled properly. The objective of this document is to illustrate our competencies in this domain.

EQRC

EQRC is Dr. Mahdavi-Damghani's limited company officially launched in 2014 in order to create a legal platform initially and primarily focused on consultancy assignments in the Quantitative Finance industry.

Normalizing Rolling Contracts

In the traditional asset classes, most listed derivatives markets typically offer new contracts once a month with a two year expiry. However few markets have shorter expires and the rules associated with the issuance of new contracts are often unnecessarily convoluted. In any case, the days in which a contract is issued, we have a new set of data points conveying refreshed information about the IVS and we can also observe old contracts current bid-ask spread. In the traditional asset classes, it is usually agreed that there exists 9 important standardized pillars in which linear interpolation in variance space gives reasonable results. These pillars are ON, 1 Week (1W), 2W, 1M, 2M, 3M, 6M, 1 Year (1Y) and 2Y. In few of the markets (eg: commodities, cryptocurrencies) none of these pillars are actively traded and they are also shorter in terms of expiry. In other words there are no standardised perpetuals as understood by traditional institutional investors.

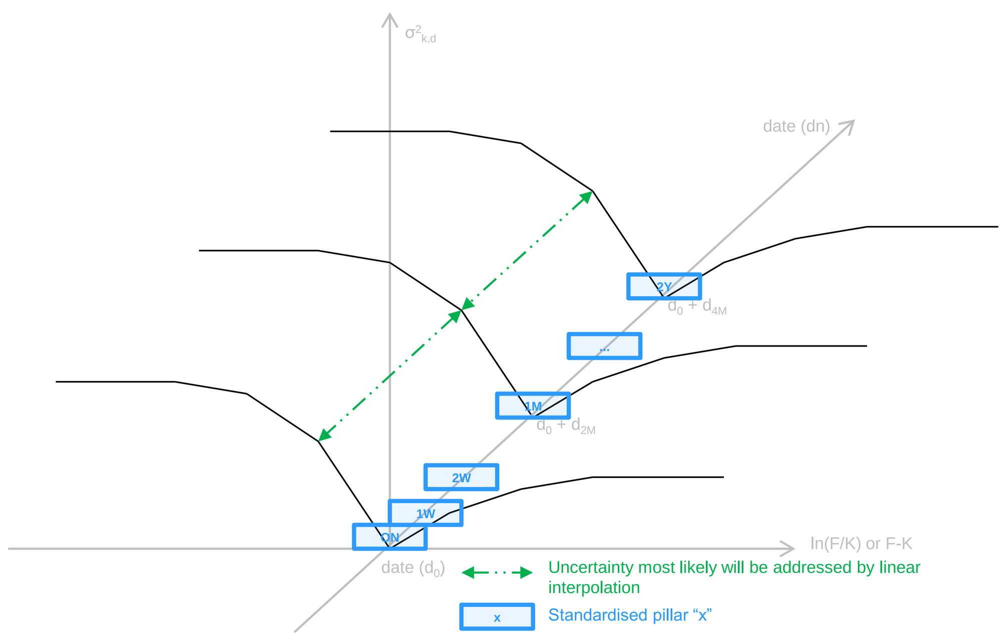

Figure 1: Example of a hypothetical IVS along the standard trading pillars.

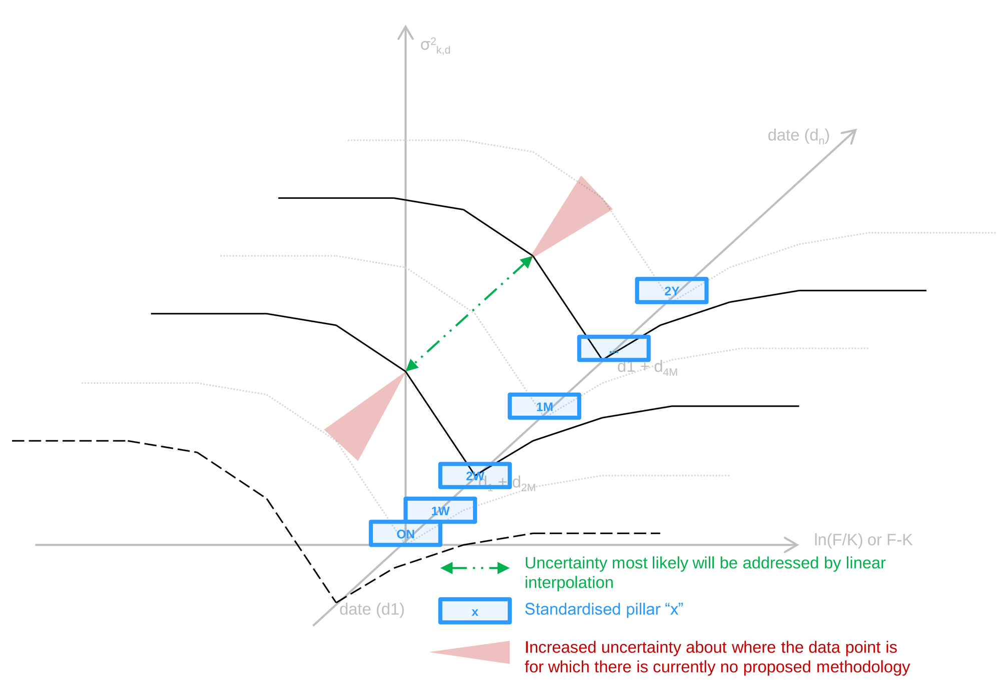

Figure 2: Example interpolation around the standard trading pillars

Another problem is that specific strikes (eg: the wings) and specific tenors are less liquid yet correlated with the ATM. The question becomes, how can we diffuse this newly arrived data onto the other points of the IVS? The answer to this question is risk factor decomposition combined with liquidity modelling while considering the usual arbitrage constraints.

Implied Volatility from Options Prices

Parameterizing the volatility surface solves many issues including but not limited to the above. As a reminder, vanilla options model suppose constant volatility with either the Black-Scholes model (BSM) or some closely related model (eg: Garman Kolhagen, Normal Distribution assumption, Black model). The constant volatility in the BSM is is an assumption that is mathematically convenient but ultimately not verified by the market. To reconcile the data with the model, we needs a correction to complete its limitation. This correction or "overlay model" is what is commonly known as the IVS. The reason why we chose to change the volatility is because it is the only non-observable value. We call implied volatility the geometrical 3D structure (e.g. figure 3 is an example) which takes as input a tenor and a moneyness and returns the volatility value which reconciles the BSM equation to the market observable price. There exists several well known numerical methods for fetching the implied volatility surface from the options market prices. For instance the Bisection method, the Newton-Raphson method, the Secant method or the Brent algorithm (their ensemble algorithm) are few of these methods.

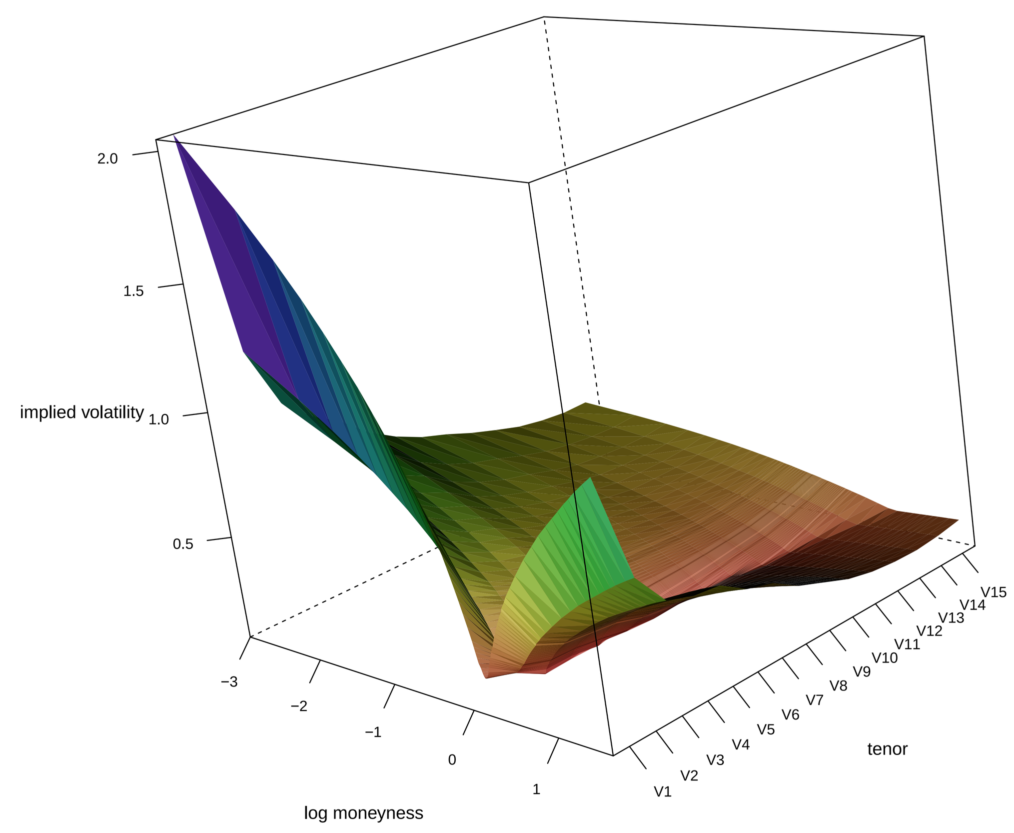

Figure 3: Arbitrage Free IVS with 15 standard tenors & 5 standard strikes

Arbitrage Conditions

Condition on the Strike Space

Theorem A Call $C(\cdot)$ is arbitrage free on the strike axis if and only if

$$\frac{\partial^2 C}{\partial K^2} = \phi(S_T, T) > 0 \tag{1}$$

Using numerical approximations we obtain equation (2)

$$\forall \Delta, C(K-\Delta) - 2C(K) + C(K+\Delta) > 0, \tag{2}$$

which is known in the industry as the arbitrage constraint of the positivity of the butterfly spread. Essentially a call of a lower strike must always be more expensive given all the other parameters constant.

Common Error

However the financial mathematics literature contains errors. Let us examine the following result.

Proposition It is incorrectly believed that

$$|T \partial_K \sigma^2(K,T)| \leq 4, \forall K, \forall T, \tag{3}$$

Counter Example We explore below a counter example.

Let us call $\chi_R = \{a, b, \rho, m, \sigma\}$ a given parameter set for the function $f : \mathbb{R}^{+,*} \times \mathbb{R}^{+} \times [-1,+1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^{+,*}$ given by

$$f(k, \chi_R) = a + b[\rho(k-m) + \sqrt{(k-m)^2 + \sigma^2}] \tag{4}$$

Lemma Applying our proposition to $f(.)$ we obtain:

$$b(1+|\rho|) \leq 4/T, \forall K, \forall T, \tag{5}$$

Proof. Simply take $\partial_K f(k, \chi_R) = b(1+|\rho|)$. Now applying proposition to $b(1+|\rho|)$ we obtain equation (5). □

Though mathematically correct a counter example exists:

$$(a, b, m, \rho, \sigma) = (0.0410, 0.1331, 0.3586, 0.3060, 0.4153). \tag{6}$$

For instance, figure 4 represents the counter example of $|T \partial_K \sigma^2(K,T)| \leq 4$ applied to $f(.)$ parametrization.

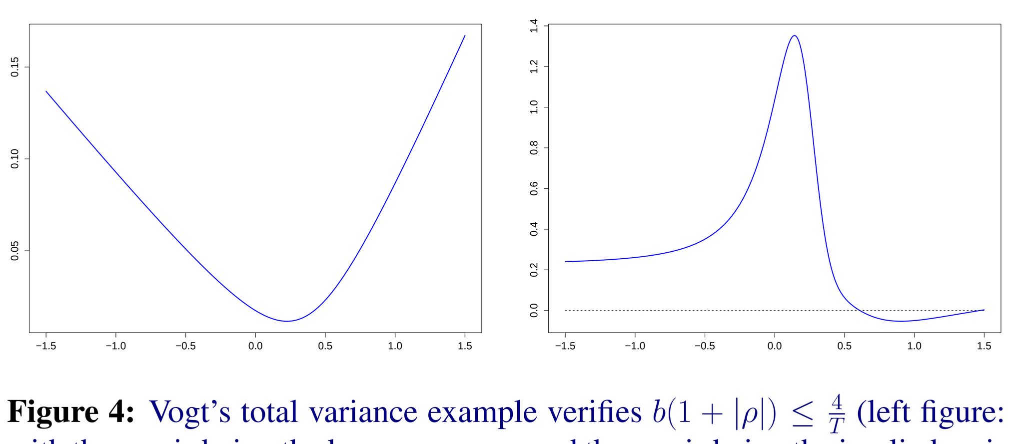

Figure 4: Vogt's total variance example verifies $b(1+|\rho|) \leq \frac{4}{T}$ (left figure: with the x axis being the log-moneyness and the y axis being the implied variance). The corresponding $\partial^2_{K,K} BS\left(\sigma^2(K,T)\right)$ approximating the (supposed) always positive pdf (right figure: with the x axis being the log-moneyness and the y axis being the non normalized pdf)

More specifically on the left hand side of the figure 4, we can observe a standard implied volatility plotted using the equation (4) with Vogt's example. The latter example verifies the constraint $|T \partial_K \sigma^2(K,T)| \leq 4$ yet, on the right hand side the PDF yields a negative value inferring an arbitrage. The main point of this example is not really the condition itself but rather that this topic is complex but made worst as a great deal of the financial literature contains mistakes that practitioners are not always aware of and which can have important buisness consequences.

Condition on the Tenor

There exists another arbitrage constraint, this time on the tenor axis. Indeed, following

Theorem Dupire's local vol model a Call $C(\cdot)$ will be arbitrage free on the tenor axis if and only if

$$C(K, T+\Delta) - C(K e^{-r\Delta}, T) \geq 0. \tag{7}$$

What this means is that an option with a longer expiry should always be more expensive given every other parameters constant.

Put Call Parity

Theorem If the Call price is given by $C(S,K) = \max(S-K)^+$ and the put price is given by $P(S,K) = \max(K-S)^+$, we have:

$$C(t) - P(t) = S(t) - K \cdot e^{-r(T-t)}. \tag{8}$$

Proof. An easy way to see why this must be true so set $r = 0$ and notice that $C - P = \max(S-K)^+ - \max(K-S)^+ = S - K$. □

However this condition is not always priced well in Binance.

Implied Correlation

There exist yet other conditions notably on the relationship between the vols of correlated assets. These conditions are sometimes known in FX as implied correlation arbitrages. They are also relevant to the cryptocurrency markets but are not as much monitored. For instance, let $S_{t,1}$ be the exchange rate of EUR/USD pair and σ_1 its implied volatility. Similarly let $S_{t,2}$ be the exchange rate of USD/JPY pair, σ_2 its implied volatility and $S_{t,3}$ be the exchange rate of EUR/JPY pair and σ_3 its implied volatility. We know that $S_{t,3}$ must be equal to $S_{t,1} \times S_{t,2}$ else the arbitrage opportunity induced would be immediately taken advantage of by HFT. Therefore $\ln(S_{t,3}) = \ln(S_{t,1} \times S_{t,2})$. Taking the variance on each side we get the non arbitrage condition on the volatility and the implied correlation given by equation (9).

$$\sigma_3^2 = \sigma_2^2 + \sigma_1^2 + 2\rho_{1,2}\sigma_1\sigma_2 \tag{9}$$

By rearranging, the implied correlation can be isolated and given by equation (10).

$$\rho_{1,2} = \frac{\sigma_3^2 - \sigma_2^2 - \sigma_1^2}{2\sigma_1\sigma_2} = \cos\phi_{1,2} \tag{10}$$

Figure 5 represents a visual representation of this non arbitrage constraints with $\phi_{1,2}$ representing the angle between σ_1 and σ_2.

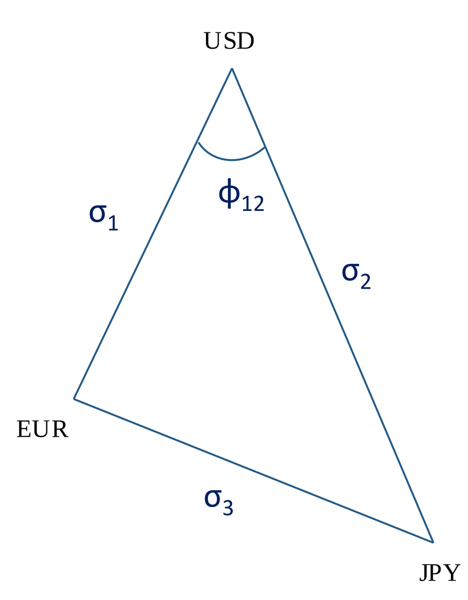

Figure 5: FX triangle

The relationship between $\rho_{1,2}$ and $\phi_{1,2}$ is given by equation (11).

$$\arccos\rho_{1,2} = \phi_{1,2} \tag{11}$$

$$\sigma_1 + \sigma_2 + \sigma_3 > 2\max(\sigma_1 + \sigma_2 + \sigma_3) \tag{12}$$

De-arbitraging

Often, a volatility surface which is constructed with an asynchronous data set and with a disregard to the bid ask spread. This makes it incoherent when it comes to the few arbitrage constraints discussed previously. There exist several pricing methodologies. However, independently from the underlying diffusion model, for our de-arbitraging methodology, we make sure that for every pillar tenor and every pillar strikes the relevant points are mutually arbitrage free. Though intuitively understandable, the optimisation by constraint methodology is famous for being unstable.

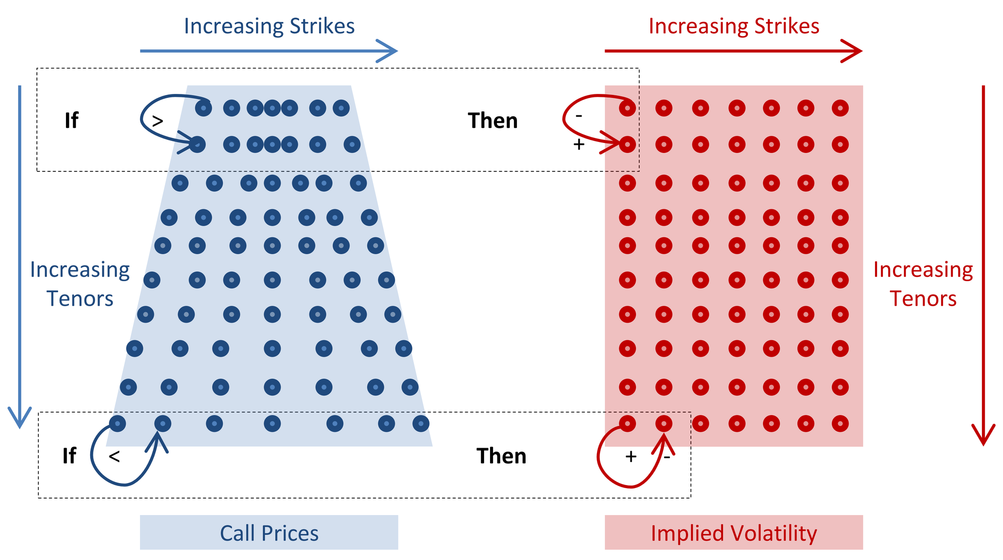

Figure 6: Visualization for the core simple de-arbing idea approximation

Note that markets such as FX and Cryptocurrencies have additional constraints (implied covariance). We also make sure that the volatility surface associated with the call and the put are the same. We have experience in de-arbitraging volatility surfaces in all the asset classes (Equities, Commodities, FX, FI and Cryptocurrencies).

Volatility Parametrization

Motivation

Volatility parametrization, a way to reduce the dimensionality of surfaces may have very different motivations depending on ones function within a financial institution. Starting from pricing and trading, volatility surfaces are also critical for margining and risk management in general. We present below our own model, the Implied Volatility surface Parametrization (IVP). The model has become a market standard so much so that it is currently taught in the CQF. The formula can be quite convoluted for a business proposal but we have had multiple presentations on the model if the reader is interested in the details (please click the thumbnail of figure 7). However we have incorporated few diagrams on how its parameters behave intuitively. The first 5 parameters are in common with the famous SVI model.

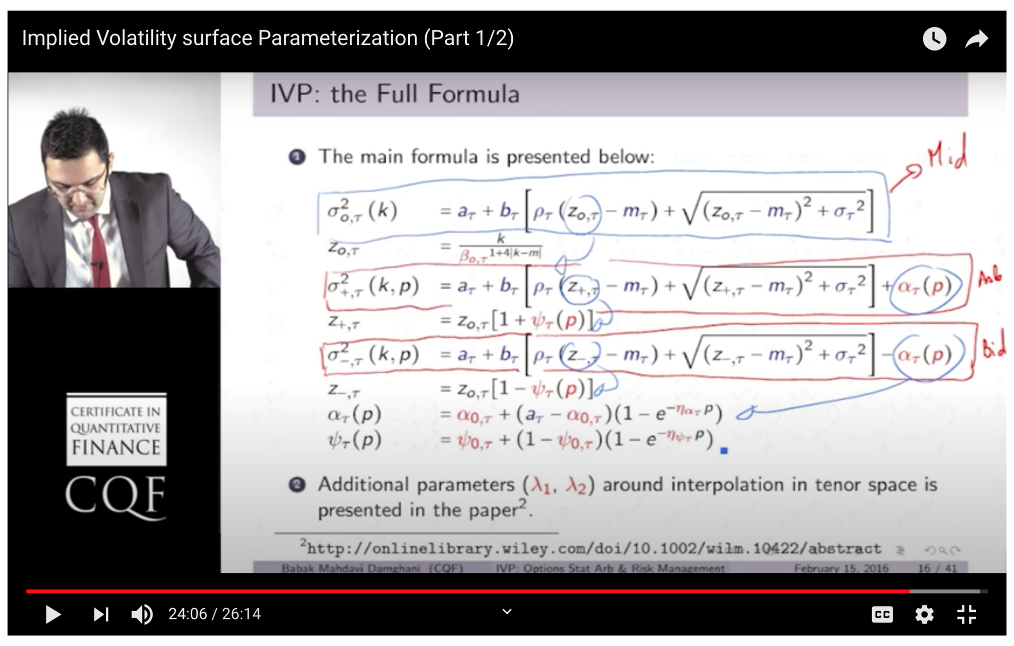

Figure 7: Clickable YouTube thumbnail linking our IVP model explaining how we plan to handle implied volatility pricing for this new illiquid market.

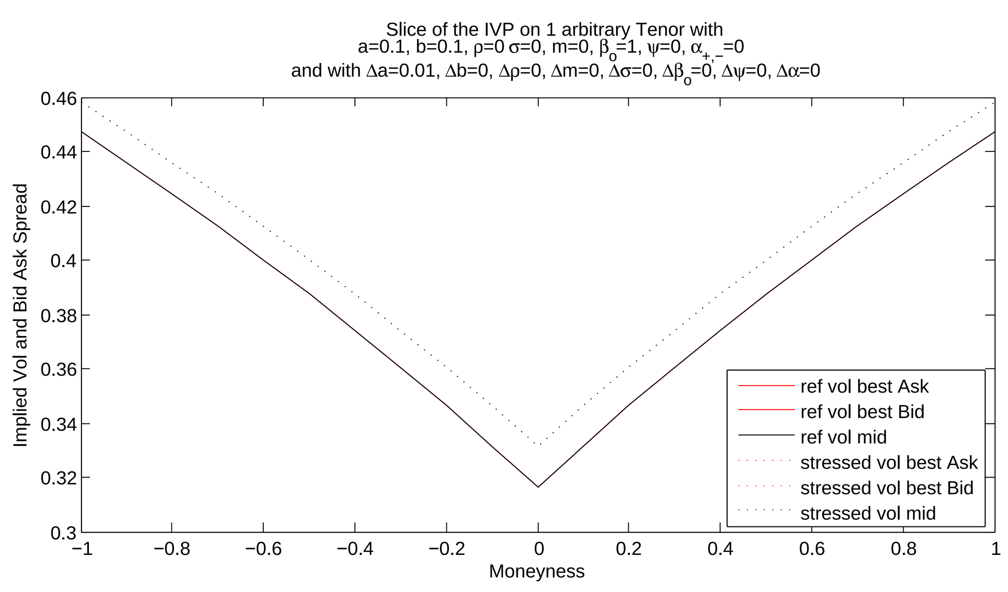

Figure 8: Impact of a change in the value of parameter a in the IVP

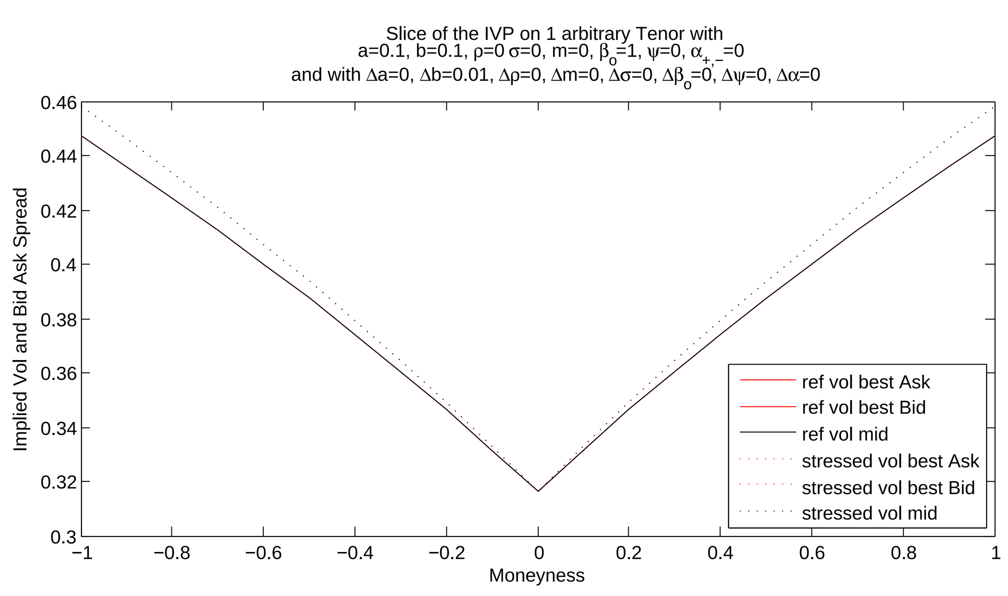

Figure 9: Impact of a change in the value of parameter b in the IVP

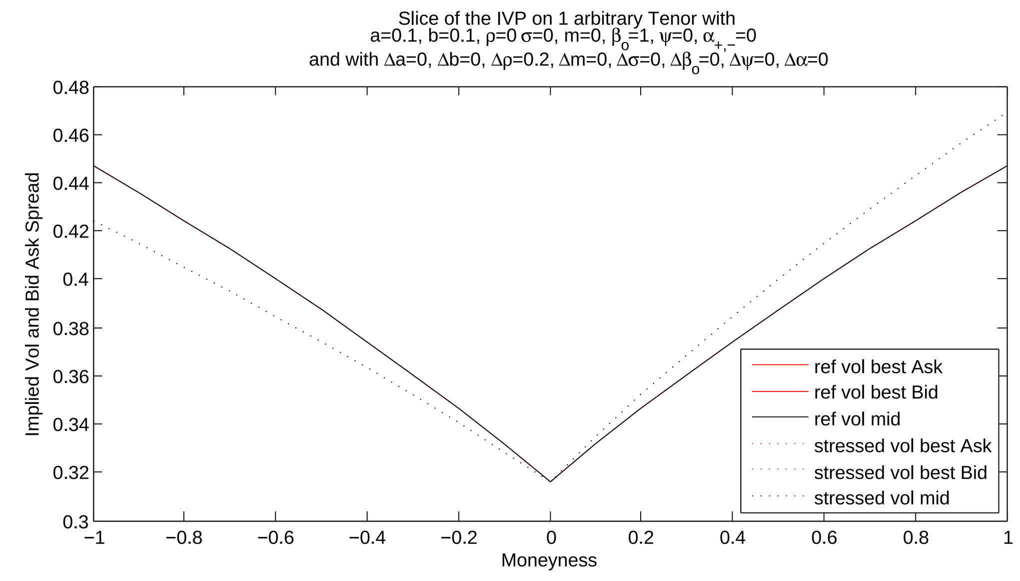

Figure 10: Impact of a change in the value of parameter ρ in the IVP

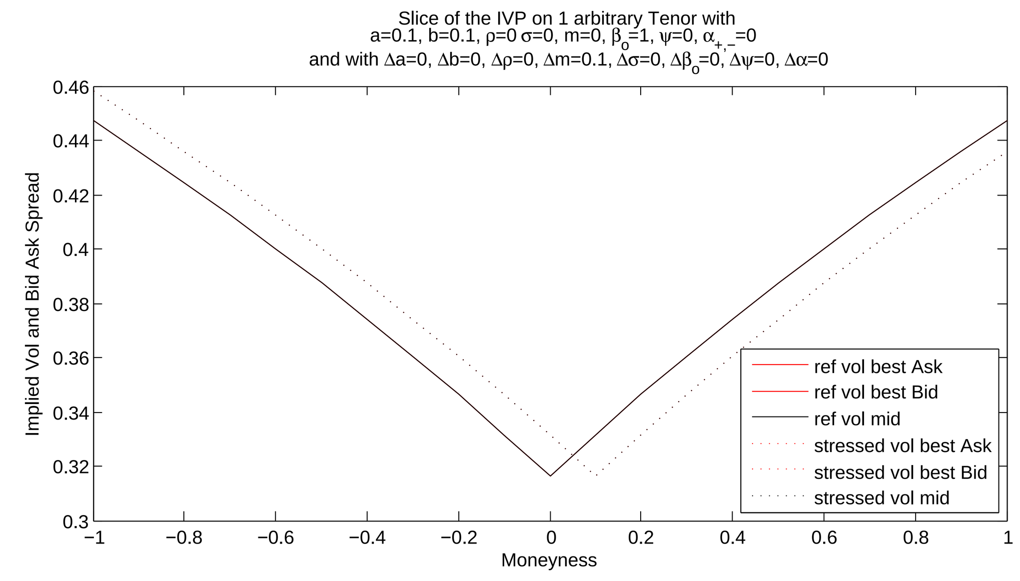

Figure 11: Impact of a change in the value of parameter m in the IVP

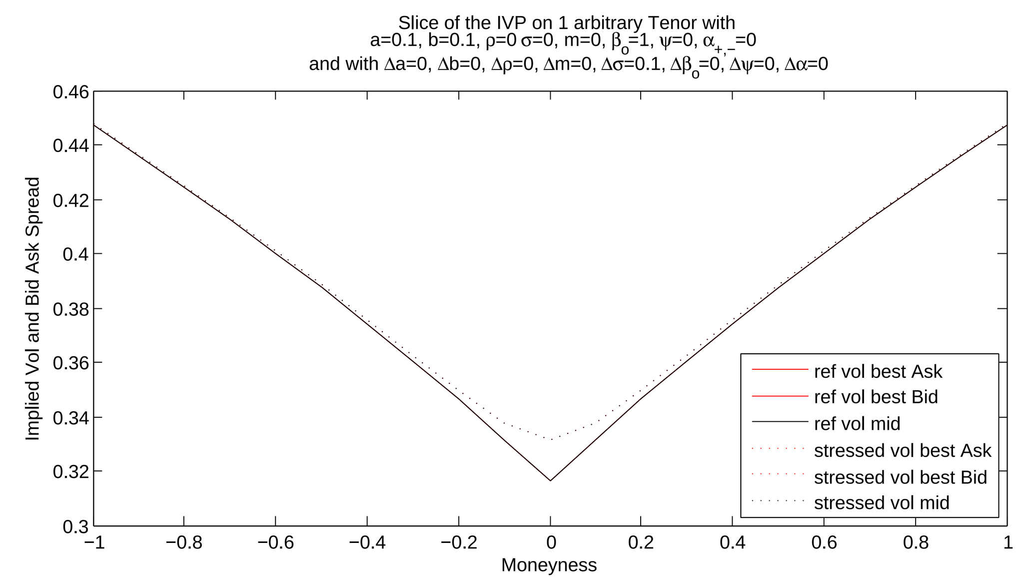

Figure 12: Impact of a change in the value of parameter σ in the IVP

The next one addresses the model limitations around the wings.

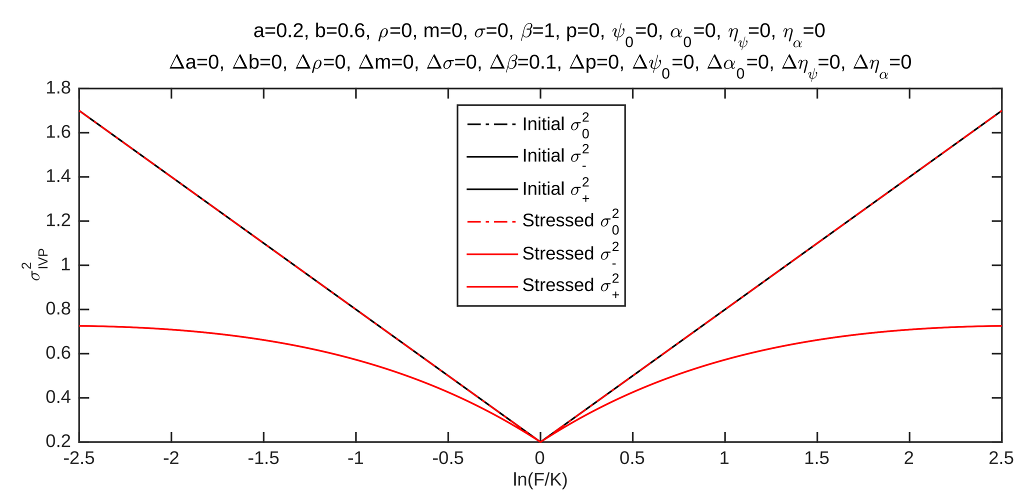

Figure 13: Impact of a change in the value of parameter β in the IVP model

The next five parameters address liquidity issues.

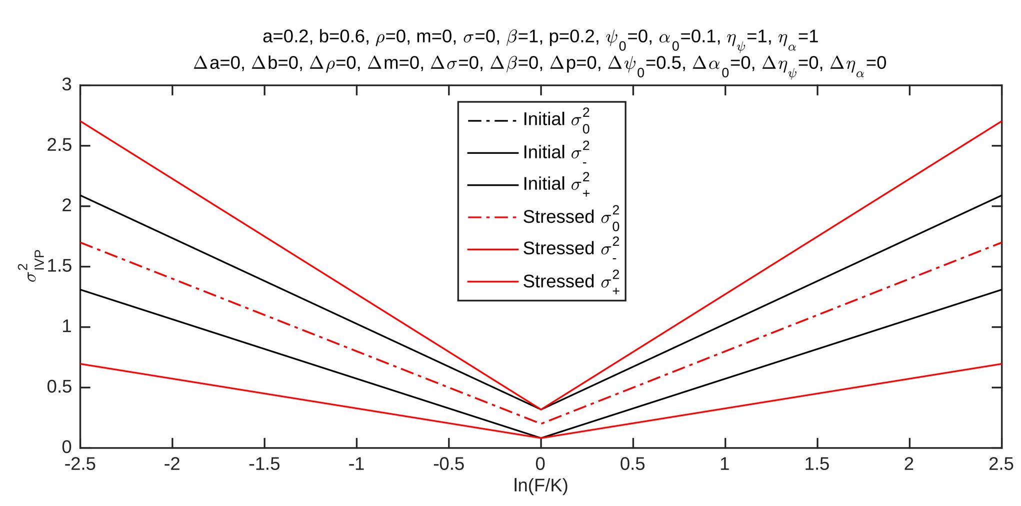

Figure 14: Impact of a change in the value of parameter ψ in the IVP model

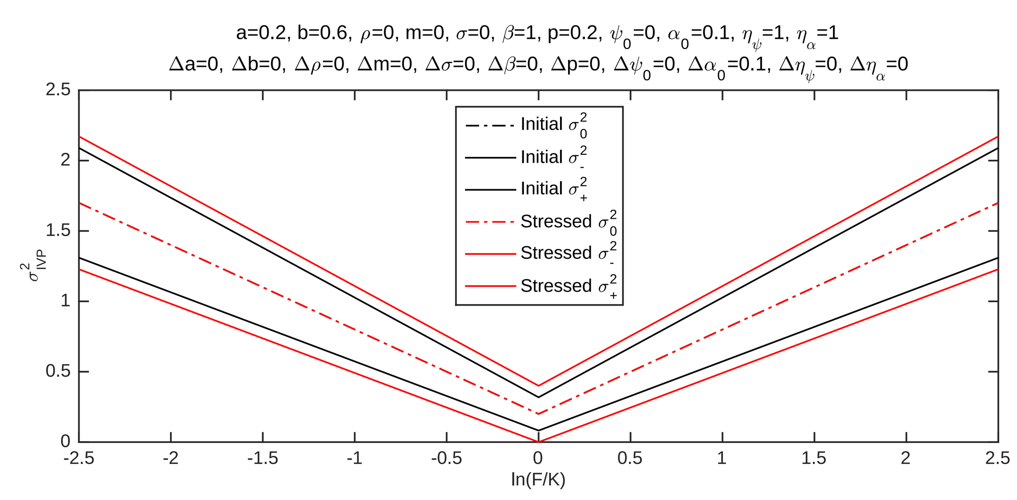

Figure 15: Impact of a change in the value of parameter α in the IVP model

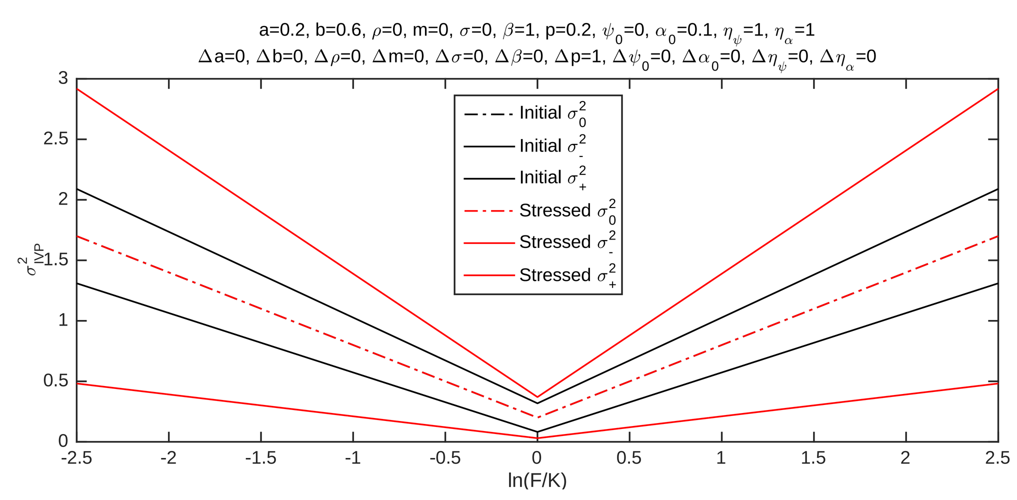

Figure 16: Impact of a change in the value of parameter p in the IVP model

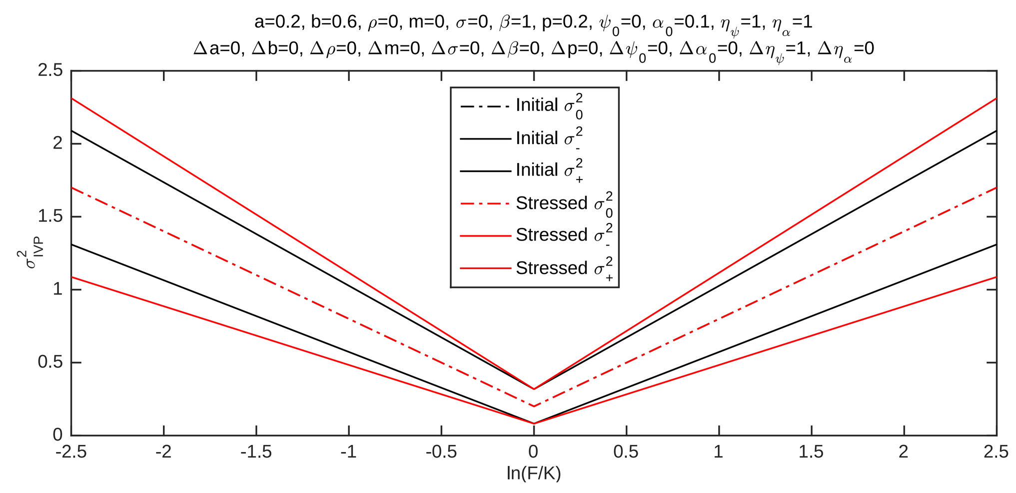

Figure 17: Impact of a change in the value of parameter η_ψ in the IVP model

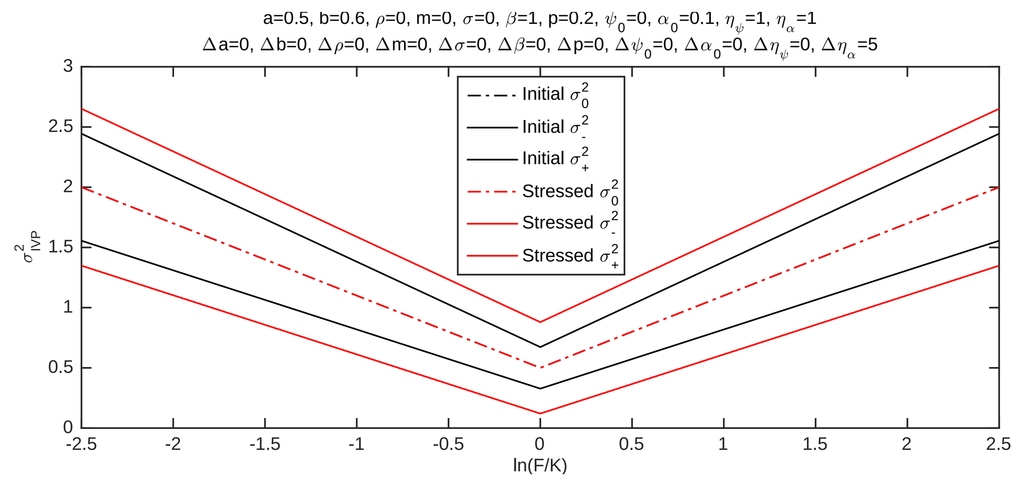

Figure 18: Impact of a change in the value of parameter η_α in the IVP model

Tracking Complex Scenarios

The question of marking the IVS in an asynchronous fashion is also a central problem for which we have a solution. For instance figure 19 represents a slice of our IVS. We can see that a random arrival at a specific point of the IVS may diffuse throughout the tranche and beyond but the way this is done is critical in the context of Market Making. for instance in figure 19 the arrival of information on a specific moneyness (the arrow) could mean many things:

1. the specific moneyness has increased without any change on the remainder of the IVS,
2. the overall IVS has increased but we only observe one specific point,
3. a change of skew has occurred,
4. a change in the total variance structure has occurred,
5. a localized change has occurred,

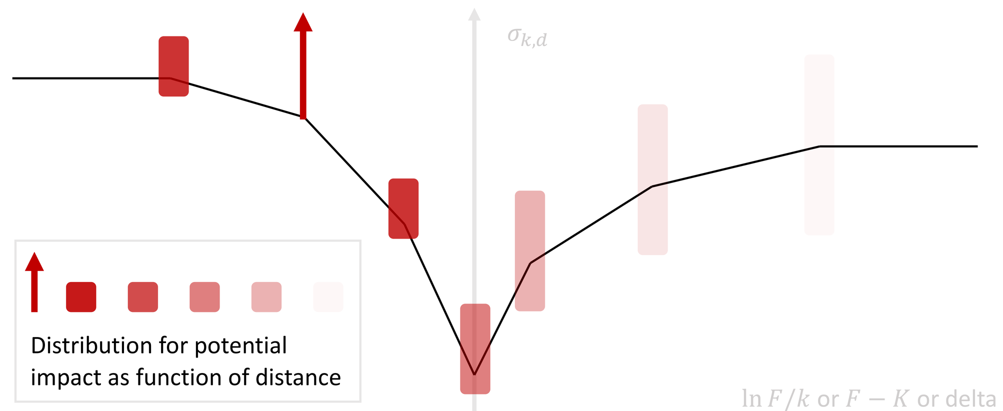

Figure 19: Asynchronous information arrival on specific moneyness and the intuitive representation for the impact on other the other strikes of the same tenor

We use our own bespoke SMC to build these complex scenarios in order to have a more holistic view of the risk associated with the IVS.

More about OCP's quant team

Dr. Babak Mahdavi-Damghani (BMD) completed his PhD in Machine Learning for Quantitative Finance at the University of Oxford. He has a broad range of work experiences in the financial industry notably having worked for Citigroup, Socgen, GAM Systematic (Cantab Capital Partners), Credit Suisse, LCH, LSEG, Oxford Man Institute of Quantitative Finance, Andurand Capital and the Oxford Algorithmic Trading Programme. He has experience through all the major asset classes in both the buy & sell sides. He is the founder of EQRC and the author of numerous publications, including cover stories of Wilmott magazine and mathematical models currently taught at the CQF (eg: Cointelation, IVP, UTOPE & HFTE model).

The **quant team** (QT), is composed of an exceptional group of professional talents selected through a long rigorous selection process involving, technical expertise, creativity and integrity. All the members of the QT have postgraduate diplomas from top tier Universities. The requirements are a specialised MSc in Quantitative Finance (or strongly related fields) or a PhD in a Quantitative field (Machine Learning, Mathematics, Physics, Computer Science, Engineering, Statistics) with relevant work experience.

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