Guidelines for Building a Realistic Algorithmic Trading Market Simulator for Backtesting while Incorporating Market Impact

Agent-Based Strategies in Neural Network Format, Ecosystem Dynamics & Detection
(working paper)

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Abstract— **In this paper, a shorter and more publication focused version of our recent article "A Bottom-Up Approach to the financial Markets" [37] is presented. More specifically we propose a new approach to studying the financial markets using the Bottom-Up approach instead of the traditional Top-Down. We achieve this shift in perspective, by re-introducing the High Frequency Trading Ecosystem (HFTE) model [38]. More specifically we specify an approach in which agents in Neural Network format designed to address the complexity demands of most common financial strategies interact through an Order-Book. We introduce in that context concepts such as the Path of Interaction in order to study our Ecosystem of strategies through time. We show how a Particle Filter methodology can then be used in order to track the market ecosystem through time. Finally, we take this opportunity to explore how to build a realistic market simulator which objective would be to test real market impact without incurring any research costs.**

Keywords: Generative Adversarial Networks (GANs), High Frequency Trading Ecosystem (HFTE), High Frequency Financial Funnel (HFFF), Multi-Target Tracking (MTT), Stability of Financial Systems, Markov Chain Monte Carlo (MCMC), Data Analysis and Patterns in Data, Electronic Trading, Systemic Risk, High Frequency Trading, Game Theory, Machine Learning, Predator Prey Models, Sequential Monte Carlo, Particle Filter.

I. Introduction

1) A Call for a Modelling Revolution: Though used sometimes loosely partly because of a lack of formal definition, the interpretation that seems to best describe Big Data is the following. The definition is really twofold. The first is one associated with a large body of information that we could not comprehend when used only in smaller amounts [14]. This characterization seems to indicate that the realm of the definition goes fundamentally beyond simply reducing the confidence interval of a parameter whose estimation would benefit from an increase of the sample size. This latter intuition is the natural statistician point of view. In fact the term "datafication" has recently been introduced in order to replace the misleading term that is Big Data in order to make sure readers research the term instead of guess its meaning [14], [37]. A good way to illustrate this point with finance would be for instance to examine Figure 1 which represents new data at the high frequency domain. The latter allows us to shift our study of the market from the Top-Down[1] approach to the Bottom-Up approach[2]. Indeed the Figure cannot be explained by the TD approach as the fluctuations seem to be more driven by systematic strategies interacting into a quagmire. The new candidate sector under inspection, after the sub-prime crisis, quickly became the one of algorithmic systematic trading which flash crash of May 6, 2010 (in which the Dow Jones Industrial Average lost almost 10% of its value in matter of minutes) exacerbated the scrutiny. However, the current state of the art risk models, are the ones inspired by the last subprime crisis and are essentially models of networks in which each node can be impacted by the connected nodes through contagion [23] and is better suited to lower frequency models. Figure 1 suggests that the common, perhaps lazy view, that crashes occur through totally unpredictable [54] events may not be true for algorithmic trading. In fact market impact literature has gained noticeable momentum in the recent past [42]. Market impact study is one of the rare areas of Quantitative Finance where interaction of strategies is taken seriously. In any case Figure 1 and the other multiple flash crashes has led the scientific community to encourage revolutionary changes to occur, possibly in the form of agent-based modelling [5], [19], [8] in lieu of traditional financial mathematics models. It is in this fundamental opposition of views that part of the title of our original paper must be understood [37].

A. Problem Formulation

1) Bottom-Up vs the Top-Down: We learn about the Bottom-Up vs the Top-Down approach in introductory systems engineering classes at the undergraduate level but by the time one gets into the most advanced postgraduate financial mathematics classes, this essential beginners level scientific lesson, has long been forgotten and the models have become dogma. Indeed at these more advance stages of ones education it becomes much more important to be able to derive or infer meaning via these believes rather than understand

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[1] e.g. the market is assumed to be a Brownian motion but allows the clever construction of dynamical strategies such as hedging for instance.

[2] e.g. strategies interacting explain the dynamics of the market

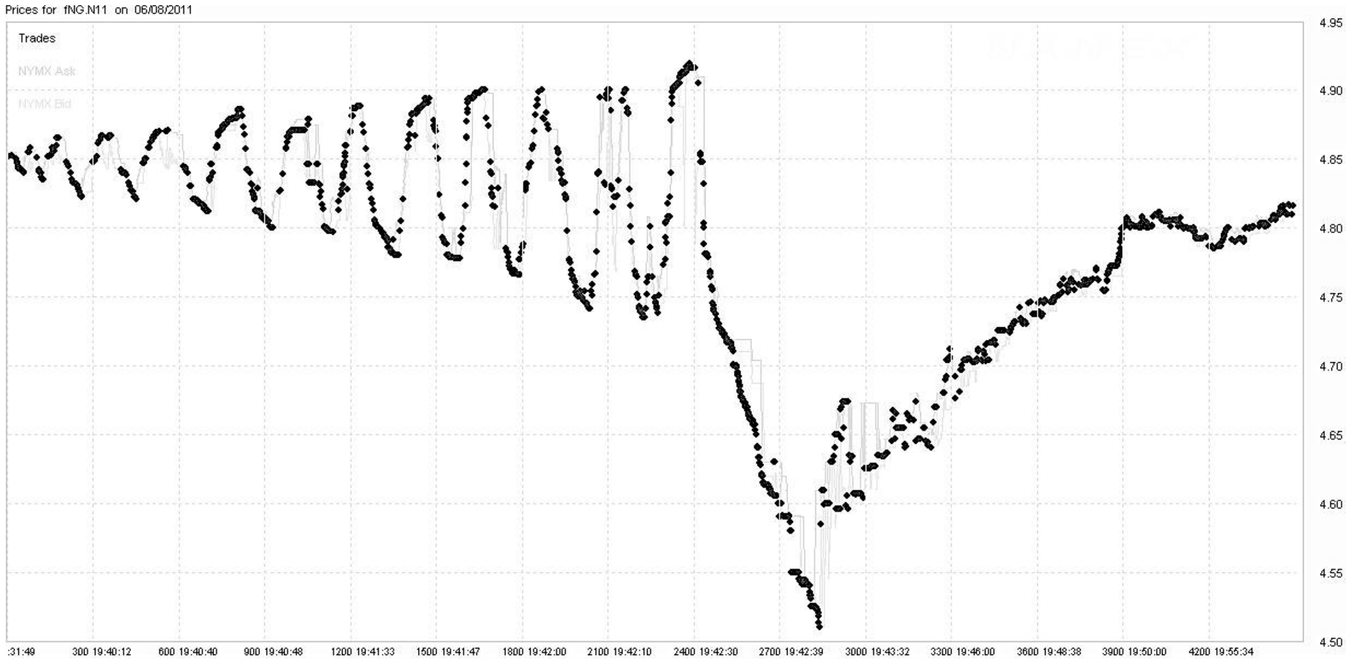


Fig. 1: Natural Gas flash crash of 06/08/2011 [45]

the limitations of these core modelling assumptions and improve the models from inception. In fact these beliefs are so much anchored in our common academic psyches that wrong models get Nobel Prizes[3] and lead to market crashes and bankruptcies[4].

2) Market Impact's Importance: It has been shown recently that, in the context of correlated assets, synchronicity is key when it comes to optimal execution [42]. There are different ways to see how this is related to the Bottom-Up approach. For us, it suggests that quantitative asset managers who employ low frequency optimal index construction centered around volatilitysuch as Risk Parity [40] or more centered around historical return[5] act as a prey for predator algorithms in the higher frequencies[6].

3) Evidence of an evolving strategy ecosystem: Being aware of their condition as "preys", asset manager have deployed defense mechanism to protect themselves from the predator algorithms. For instance in a situation in which an asset manager needs to re-balance its portfolio in a sequential manner, then HFT act on the correlated assets before asset managers (at the lower frequency) had time to balance completely their portfolio. This has naturally created the need to synchronize the re-balancing execution process. If we were to take the situation in it historical context and perhaps re-introduce the concept of invasion into a strategy ecosystem, then we can say that the Robot Ahistorical asset manager systematic trading which did not employ synchronicity was historically acting as a prey to Robot B[7] which led the asset managers who had designed Robot A to create now a Robot C[8]. So we can see trading environments changing as a result of the frequency of certain strategies changing. Though interesting these hypothesis need to be understood more deeply, going back to the most simple infinitesimal granularity so to explain the extraordinary complexity into well understood incremental steps. Once these incremental steps are understood, can we apply our new understanding to a more practical business application? More specifically can we use this Bottom-Up approach to build a realistic market simulator[9]?

B. Agenda

We have divided this paper in 3 Sections. More specifically in Section II we express some agent-based strategies in Neural Network format suggesting the incentive for going building complexity in simple steps going from Shallow to Deep Learning. We will use the HFFF format [38] recently introduced for this exercise, summarize some classic strategies as well as suggesting un-intuitive ones. This will then help us, in Section II, to introduce a couple of methodologies for studying Ecosystems of strategies through time using tools in evolutionary dynamics and defining more rigorously the concept of regime change. The first method, the brute force genetic algorithm will be contrasted with the Path of Interaction methodology which will serve us in the last section, IV, in which we study Particle Filter methods

[3]see: Black-Scholes model.

[4]e.g. Long-Term Capital Management history.

[5]such as Fernholz' model [20]

[6]such as High Frequency Trading algorithms that employ statistical arbitrage opportunities based on movements of the order book.

[7]HFT strategy that would quickly buy correlated assets upon seeing a big market movement: the basis of statistical arbitrage.

[8]e.g. Risk Parity but with synchronicity in terms of execution strategy

[9]This is important because we would like to build a system in which we can test market impact without real money.

applied to Multi-Target Tracking (MTT). We will finish this section by providing guidelines on how to build a realistic market simulator, leveraging on the material introduced in the previous sections.

II. Agent-Based Intelligent System, from Shallow to Deep Learning

Following Bouchaud[10]'s call for a revolutionary change in economics [5] via Agent-Based Models, this section will focus on the construction of these agents. More specifically we summarize some of the agents already introduced in our last paper [38] but also new agents we thought were interesting to discuss.

A. Electronic Trading

Traditional order book [10] consists of a list of orders that a trading venue such as an exchanges uses to record the market participants' interests in a particular financial product. Typically a rule based algorithm records these interests taking into account, the price and the volume proposed (on either side of the Bid-Ask) as well as the time in which that interest was recorded (in situations in which interest at the same price is recorded by few different market participants, a referee decides which would win the trade: usually FIFO).

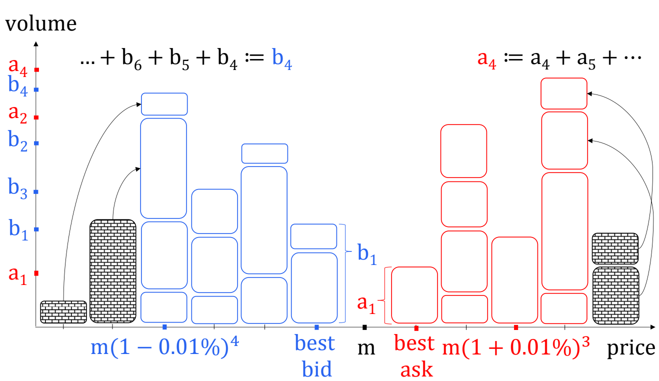


Fig. 2: Order-book visual representation

Definition (Order-Book): In terms of naming early these different points of the order book we would label by a_1^t and b_1^t the best ask & bid total volumes at time t. By extension a_i^t, b_i^t with $i \in \{1, 2, 3, 4\}$ would correspond to total volume at the relevant depths' of the order book with the special case where $i = 4$ which then would represents the total volume at the 4th depth level in addition to all the other market depths superior in price (in the case of the Asked price and vice versa for the bid price). We will call m_t the mid price of the best bid/ask at time t. The prices at the different levels, l of the order book will be arbitrarily chosen to be 1bps[11] apart as shown by Equation (1). Figure 2 represents our version of the order book.

$$p_l^t = m_t[1 + (-1 \times 1_{l \in b_i^t} + 1 \times 1_{l \in a_i^t}) \times 0.001\%]^i \qquad (1)$$

[10]Quant of the year 2017.

[11]bps stands for Basis Points or in terms of Percentage 0.01%.

Definition (Leading Indicators): We will label by $\{y_i\}_{i=0}^{n-1}$ the price process of interest, $i \in [0, n]$ its discretized 500ms snapshots with $i = 0$ being the most recent snapshot and $i = n$ its most distant snapshot. Moreover we will assume here that 500ms is enough time for the trading system to take the data, reformat it, analyze it as well allow the relevant strategy to take actions[12]. Similarly we will define $\{x_{j,1}, x_{j,2}, \dots x_{j,p}\}_{j=i+1}^{n}$ the relevant, p leading indicators to the price dynamic of interest.

Remark We will assume that the Leading Indicators for the price process can only be taken from the order book which is a reasonable assumption in the higher frequencies. Some usually accepted leading indicator are the price of the underlier itself and the accumulated volume at different market depth of the order books (4 on the bid side and 4 on the ask side for a total of 9 leading indicators with the price process: see Figure 2 for visual representation).

B. Neural Net Architecture & Learning Potential

1) A Brief Qualitative History: In the spirit of explaining the complex through simple incremental steps, like in Econophysics [6], this particular subsection is dedicated to how complexifying simple Artificial Neural Network's (ANN) architecture in depth (with more hidden layers[13]) or width (with more connection on the same layer) can lead to useful enhanced learning potential such as the one offered by Deep Learning (DL). More specifically, taking this approach can allow us to move slowly towards Deep Learning while unweaving the black box associated to the latter perplexing concept. With this in mind, two well known, but important milestones in Machine Learning are worth reminding of. Especially for the beginners, these two milestones can shed light on why the core building blocks of our HFTE model is built this way. First, Warren McCulloch and Walter Pitts [47] introduced their threshold logic model in 1943 which is agreed to have guided the research in NN for more or less a decade. Second, Rosenblatt [48], formally introduced the perceptron concept in 1962 though some early stage work had started in the 1950s. The idea of the perceptron was one in which the two inputs could act as separators[14] and therefore a direct equivalence could be made to the Multi-Linear Regression (MLR) which we will elaborate more in details is Section II-C.3. One observed limitation of the perceptron as described by Rosenblatt, in 1969, was that a simple yet critical well known functions such as the XOR function could not be modeled [44]. This resulted in a loss of interest in the field until it was shown that a Feedforward Artificial Neural Network (ANN) with two or more layers could in fact model these functions. Added, to this we have

[12]Last assumption we will make is that no slippage or other man made errors can bias our results.

[13]"Deep Learning" is arguably just a fancy word for a Perceptron with many hidden layers.

[14]the exact research was one in which the methodology acted as a $1, 0$ through a logistic activation function $f(x) = \frac{1}{1+e^{-x}}$ as opposed to a linear one. However that small distinction is not significant enough in this context to delve too much into it but deserved a clarification in the footnotes.

the well known over-fitting [53] problems when it comes to supervised learning which, to some extend would like to simply keep adding hidden layers when the learning potential has been absorbed. This problem of learning potential to over-fitting has been there since inception though regular progress is being made in that domain without real breakthrough[15]. A real breakthrough happens however in 1956 with what is now know the Kolmogorovs superposition theorem [31] which we formalize next.

2) Kolmogorov-Arnolds Superposition Theorem: Born in what is speculated as a heated supervisor/supervise relation[16], the Kolmogorov-Arnolds superposition theorem [31], [1] is perhaps the most remarkable result in formalized mathematical machine learning of the 20th century. It states that every multivariate continuous function can be represented as a superposition of continuous functions of two variables. First designed to address Hilbert's thirteenth problem that he presented in Paris in a mathematics conference in 1900, the theorem ended up being a generalizationThe initial problem was to solve 7th degree equation using algebraic continuous functions of two parameters. of what was considered one of the top 23 most important problem as defined by Hilbert. More formally if f is a multivariate continuous function, then f can be written as a finite composition of continuous functions of a single variable and the addition [31].

Superposition Theorem: Let $f : \mathcal{I}^n := [0,1]^n \to \mathcal{R}$ be an arbitrary multivariate continuous function. Then it has the representation

$$f(\mathbf{x}) = f(x_1, ..., x_n) = \sum_{q=0}^{2n} \Phi_q \left(\sum_{p=1}^{n} \phi_{q,p}(x_p) \right) \quad (2)$$

with continuous one-dimensional inner and outer functions Φ_q and $\phi_{q,p}$. All these functions Φ_q, $\phi_{q,p}$ are defined on the real line. The inner functions $\phi_{q,p}$ are independent of the function f[17].

The impact of the theorem prompted several contributions which can be roughly organized in waves. The first of his waves was focused on the inner and outer functions of the theorem. Notably, Lorentz relaxed the constraint on the outer functions Φ_q and noticed that they could be the same [36], [35]. Sprecher proved that the inner functions $\phi_{q,p}$ can be replaced by $\lambda_p \phi_{q,p}$ with some rules around the λ_p [51], [52]. The second wave focuses on the results of this first wave and are mostly technicality extracted from mathematical analysis more specifically around the details of these inner and outer functions. The most relevant wave for us came with Hecht-Nielsen where the interpretation of the theorem was translated in a feed-forward network with an input layer, one hidden layer and an output layer [27], [28], [51].

[15]We refer here the reader to area of ML known as Regularization.

[16]Kolmogorov and Arnold published separately their results [31], [1].

[17]The full proof and potential minor improvement of the latter can be found in [7].

C. Intelligent Agents & Financial Strategies

In this subsection we will use some of the material presented in the High Frequency Trading Ecosystem (HFTE) [38] recently introduced, we will therefore summarize the main points of the referred paper for that occasion.

1) The High Frequency Financial Funnel: The pillars associated to the construction of the HFTE model has in its inspirational roots the idea that strategies in the market interact, or to chose an alternative jargon "Mutually Excite" [11], and it is their interaction that creates the fluctuations in the prices (the same way interaction create complexity in the Game of Life [21]). It also assumes that strategies can invade others and therefore the study of the financial market partially comes to studying a stochastic n-species predator prey model. Another pillar is that the construction of each of these strategies must have the same DNA[18]: the financial funnel (Figure 3). Finally the financial funnel can model many of the classic financial strategies. For example it can model Trend Following (TF) strategies, Moving Average Convergence Divergence (MACD), Multi-Linear Regression (MLR) or XOR like strategies. These few historical rationals[19] are the main drivers which have led us to propose the Funnel, introduced by Martin Nowak [46], as the simplest possible network to model (therefore which minimizes over-fitting) the key functions for our application. The area of evolutionary graph theory is quite rich. Many graphs provide interesting properties.

Definition (High Frequency Financial Funnel): We can formalize the learning process from all of our strategies using the HFFF of Figure 3 by providing a set $\mathcal{H}$, as described by Equation (3) of weights corresponding to all the possible weights of this particular figure.

$$\mathcal{H} \triangleq \left\{ \begin{array}{cc} \cup_{j\in[1,9]} w^i_{\bar{s},j} & \cup_{j\in[1,9]} w^i_{s,j}, \\ \cup_{j\in[1,9], i\in[1,3]} w^{h_1}_{\bar{s},i,j} & \cup_{j\in[1,9], i\in[1,3]} w^{h_1}_{s,i,j}, \\ \cup_{j\in[1,3]} w^{h_2}_{\bar{s},j} & \cup_{j\in[1,3]} w^{h_2}_{s,j}, \\ w^o_{\bar{s},j\in[1,9]} & w^o_{s,j\in[1,9]} \end{array} \right\} \quad (3)$$

with w^i, w^h and w^o, respectively the weights associated to the input, hidden and output layers. More formally let the **H**igh **F**requency **F**inancial **F**unnel (HFFF) [38] to be a NN of 9 inputs, 3 hidden layers and 1 output layer. Each node connects to the next layer and to itself. Each connection to itself will be label by w_s and the others by $w_{\bar{s}}$. We will admit that $w_{\bar{s}} \sim \mathcal{U}[-1,1]$ and that $w_s \sim \mathcal{U}[0,1]$ and therefore the results from Equation (4).

$$w_x \sim \mathcal{U}[-1_{x=\bar{s}}, 1] \quad (4)$$

Remark Note that in the context of this paper we have chosen to work with Martin Nowak's [46] funnel, which modification is described in Figure 3. This NN structure offers the advantage of linking some interesting bridges

[18]alternatively called HFFF or Neural Network.

[19]we will discuss more in details the Bias-Variance Dilemma in Section III-A.1.

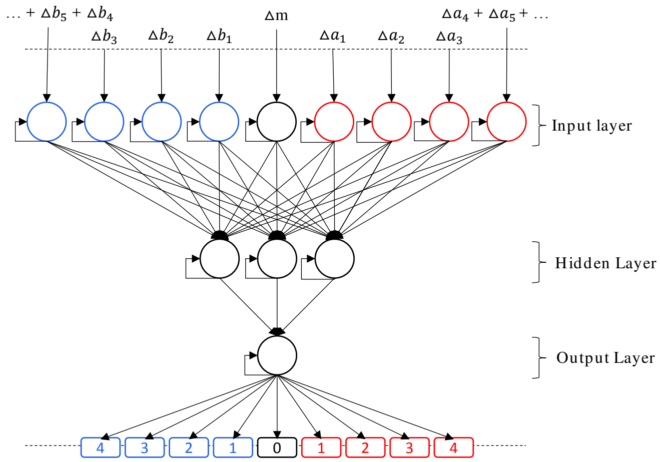


Fig. 3: The High Frequency Financial Funnel (HFFF)

between the worlds of information theory, evolutionary dynamics and biology. Indeed in information theory it also resembles the classic structure of a Neural Network and can therefore easily accommodate the mapping of classic and less classic financial strategies. In evolutionary dynamics, Moran likes Processes can easily be formalized through similar means. In biology the network structure is a potent amplifier of selection [46].

Note also that the HFFF from Figure 3 can easily be trained using a classic error back propagation algorithm like the one described in algorithm (1)[20].

Algorithm 1: Backpropagation

Input: NN $\mathcal{H}$ with unoptimized weights
Output: NN $\mathcal{H}$ with optimized weights
for *d in data* **do**

Forwards Pass:
Starting from the input layer, do a forward pass trough the network, computing the activities of the neurons at each layer.

Backwards Pass
Compute the derivatives of the error function with respect to the output layer activities **for** *layer in layers* **do**

Compute the derivatives of the error function with respect to the inputs of the upper layer neurons Compute the derivatives of the error function with respect to the weights between the outer layer and the layer below Compute the derivatives of the error function with respect to the activities of the layer below

Updates the weights.

[20]where the activation function would be linear so as to make sure the MLR strategy can be exactly replicated.

2) The EWMA NN: When we first started our research we called this subsection the Trend Following HFFF but through the simulation exercises and with increased research experience we decided to rename this subsection EWMA NN. However, in many of our simulation when we refer to the TF strategies we really mean EWMA family. We will explain this rational next. A very common trading strategy is the trend following (TF). The idea of the TF is that if the price has been going a certain way (e.g. up or down) in the recent past, then it is more likely to follow the same trend in the immediate future.

Definition (Trend Following): The mathematical formulation of a TF can be diverse but in the context of this paper we will be using an exponentially weighted moving average (EWMA) formally described by Equation (5).

$$\hat{x}_t = (1-\lambda)x_t + \lambda\hat{x}_{t-1},\ \ \lambda \in [0,1] \tag{5}$$

Remark In this equation λ represents the smoothness parameter with $\lambda \in [0,1]$. The lower the λ, the more the next move will be conditional to the immediately adjacent previous move. Conversely, the higher the λ, the more the future move will be function to the long term trend. The idea being that through a simple filtering process, the noise is extracted from the signal which then return a clean time series $\hat{x}_t$ traders like to seldom use directly or sometimes by using it with couple of other similar equations with a different λ and therefore defining a signal as a difference of these various filtered time series.

Proposition The HFFF can model trend following strategies.

We refer to our previous work [38], [37] for the proof and the diagrams. One of the current hurdles in our research is our classification issue and the MACD strategy is a good example as to why. Indeed the MACD strategy which is technically associated to the EWMA family has an economical meaning which can potentially be classified as an antithetic TF strategies (which are in the EWMA family). This may be important for practitioners as the MACD(12,26) has for instance gained a great deal of momentum for algo traders as it can be seen on the various search results on youtube or on practitioners websites such as “investopedia” [25].

Proposition The HFFF can model MACD strategies.

The Moving Average Convergence/Divergence (MACD) was designed to reveal changes in the direction and duration of a trend. It essentially models difference between a “fast” ($S_t^{N_f}$) EMA and another “slower” ($S_t^{N_s}$). For instance the popular MACD(12,26), $M_t^{12,26}$ is given by:

$$M_t^{N_f,N_s} = S_t^{N_f} - S_t^{N_s} \tag{6}$$

$$S_t^{\alpha} = \begin{cases} S_1, t=1 \\ \alpha \cdot S_t + (1-\alpha)\cdot S_{t-1}^{\alpha}, t>1 \end{cases} \tag{7}$$

$$\alpha = 2/(N_\alpha + 1) \tag{8}$$

$$N_\alpha = \{N_f, N_s\} = \{12, 26\} \tag{9}$$

Figure 4 represents a generic MACD. If one is looking specifically for a MACD(12,26), then the weights of the hidden layers must be such that $\alpha_{12} = 2/13$ and $\alpha_{26} = 2/27$ and the ones of the output layers must be a simple subtraction to abide by the above definition.

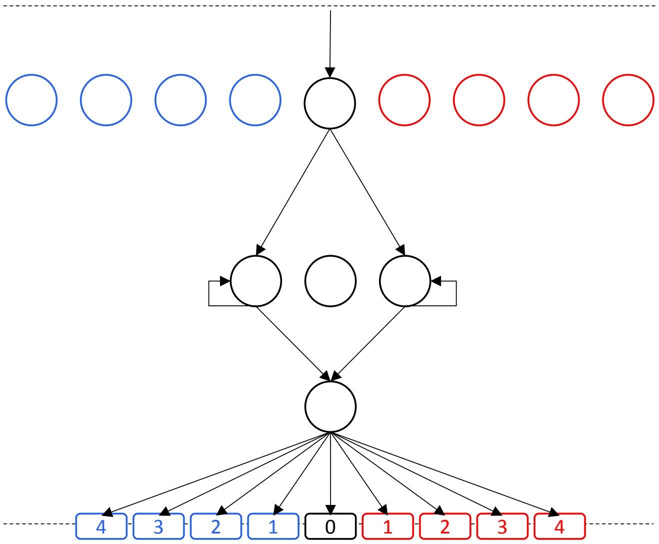


Fig. 4: MACD (difference of EWMAs) in HFFF format

3) Multi Linear Regression NN: The Multi Linear Regression (MLR) is another well known simple strategy traders have been using in the industry.

Definition (Multi Linear Regression): Given a data set $\{y_i, x_{i-1,1}, \ldots, x_{i-1,9}\}_{i=1}^n$ where n is the sample size, and y_i then our MLR is formalized by the Equation (10).

$$\begin{aligned} y_i &= \beta_1 x_{i-1,1} + \cdots + \beta_9 x_{i-1,9} + \varepsilon_i \\ &= \mathbf{x}_{i-1}^{\mathrm{T}} \beta + \varepsilon_i, \; i = 1, \ldots, n \end{aligned} \tag{10}$$

where T denotes the transpose, so that $\mathbf{x}_{i-1}^{\mathrm{T}}\beta$ is the inner product between vectors x_i and β. The best unbiased estimator of β is given by $\hat{\beta} = (\mathbf{x}^{\mathrm{T}}\mathbf{x})^{-1}\mathbf{x}^{\mathrm{T}}y$ and sometimes also referred to β^{OLS}.

Proposition The HFFF can model multi linear regression like strategies.

Remark We refer to our previous work [38], [37] for the proof and the diagrams. Do the activation functions matter when it comes to modelling MLR strategies? The answer to this question is obviously yes. The MLR is by definition a linear strategy and not a sigmoid strategy, otherwise it would be called a MSR.

4) Regularized NN & Lasso Regression: The bias-variance dilemma (BVD) is a technical term representing the optimization by constraints problem which aims at simultaneously minimizing the error from erroneous assumptions (bias) in our learning algorithm or commonly called "under-fitting" and the error from the out of sample analysis (variance) or commonly called "over-fitting". One of the properties of DL is its dual ability to learn the most complicated functions but also makes it prone for over-fitting. It is therefore recommended that one applies conscious efforts in studying carefully the associated benefits to complexity ratio in the context of the BVD. Regularization is usually the term employed for the methodology that aims at finding the optimal model according to the BVD. The mathematical formalization suggests that we calibrate a function f which takes as input a potential infinite number of explanatory variable $x_1, x_2, ...x_n$ so as to minimize the distance to a target y under some cost measure V subject to a penalization, or regularization term[21] $R(f)$. Equation (11) refers to this generic Regularization.

$$\min_f \sum_{i=1}^{n} V(f(x_i), y_i) + \lambda R(f) \tag{11}$$

Within the family of Regularized methodologies the Lasso[22] methodology is the most common one and usually associated with the MLR we have seen in the previous paragraph . They have been gaining momentum in the past few years as they represent the simplest ML technique which has the reputation to work in systematic trading provided the strategy and the input variables are sound.

Definition (Lasso Regression): Given a data set $\{y_i, x_{i-1,1}, \ldots, x_{i-1,9}\}_{i=1}^n$ where n is the sample size, and y_i then our Lasso Regression is formalized by Equation (12).

$$y_i = \beta_1 x_{i-1,1} + \cdots + \beta_9 x_{i-1,9} + \varepsilon_i. \tag{12}$$

subject to $\sum_{j=1}^{9} |\beta_j| \leqslant t$ where t is an input parameter that determines the amount of regularisation desired.

Proposition The HFFF can model Lasso regression like strategies.

Proof: Simply set $w_{\bar{s},2}^h = 0$, make sure the regularizarion is done exclusively on one of the remaining hidden layer and finally make sure the remaining hidden layer calibrates its weight the same way at the β^{OLS}. Figure 5 gives an illustration. $\square$

D. XOR Architecture

We recall here the truth table associated by the XOR function in table I. Let's look at the following known HF rational, which will hopefully shed light on the reason why we are discussing the XOR function.

I_1	I_2	O	Price (I_1)	Open Interest (I_2)	Signal (O)
1	1	0	Rising	Rising	Buy
1	0	1	Rising	Falling	Sell
0	1	1	Falling	Rising	Sell
0	0	0	Falling	Falling	Buy

TABLE I: Relationship Between Open Interest, Price & XOR

[21] or regularizer
[22] Short for Least Absolute Shrinkage and Selection Operator.

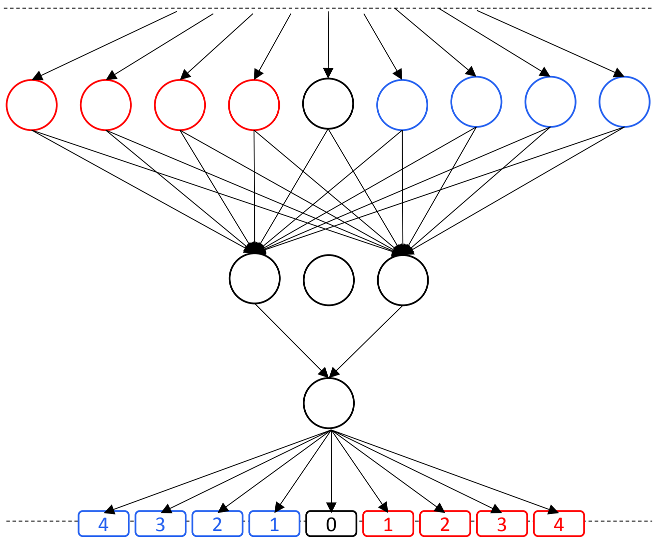


Fig. 5: Lasso regression strategy in HFFF format

Definition (Open Interest): If we define the Open Interest (OI) as being the total volume left on the order book then it is known that when the price and the OI are rising then the market is bullish, when the Price is rising but the Open Interest is Falling then the market is bearish, when the Price is falling but the Open Interest is rising then the market is bearish, and finally when the Price is falling and the Open Interest is falling then the market is bullish. These 4 market situations can be summarized by table I.

Proposition The HFFF can model XOR like strategies.

We refer to our previous work [38], [37] for the proof and the diagrams.

III. Ecosystem Dynamics

A. *Lessons learnt in our recent research*

1) Network HFFF & Deep Learning Thoughts: The scientific methodology behind the construction of the game of GO is one we wished to apply to our HFFFs and created a dynamical ecosystem. For instance increasingly advanced strategies compete with each other and we eventually get an interesting portfolio of strategies as well as their co-evolution. However, the HFFF itself potentially suffers from similar kind of limitations that prevented the XOR function to be learnt without 1 hidden layer. Indeed a legitimate question can be asked on whether a single hidden layer is enough. The answer to this question is in fact negative as Convolutional Neural Network (CNN) have shown more potential extracting trading signal compared to shallow learning [55][23]. Some other studies reveal universal features of price formation using Deep Neural Networks [50] but lack a study on simpler benchmarks (e.g: Shallow Learning). For instance in [50] a logistic regression is used for a benchmark. It would have been interesting to see some more complex benchmarks[24]. We have arbitrarily taken as hypothesis the HFFF to be good enough to model few critical strategies in the domain of QF and above all proceeding this way is important in unweaving the black box associated DL. With this in mind it is interesting to notice that the TF strategy has been designed to dominate a random swarm of strategies. In turn the MLR strategy has been designed to theoretically dominate the TF with the key point being that the MLR strategy capitalizes on areas of the orderbook the TF strategy does not have the DNA for (to perceive information of the OI). Similarly the XOR strategy has been designed to theoretically dominate the MLR by splitting the OI surface in additional zones that the MLR cannot understand (lacking the necessary hidden layer). Taking the argument forward, we could lay the hypothesis that this would eventually lead to a Farmer like strategy. The latter would consist of a highly sophisticated strategy that would understand its own impact to the ecosystem and would be able to take actions in it so as to both create long term stability as well as profit. An alternative hypothesis would be that Neural Network complexity would not matter beyond an XOR strategy and that we could eventually converge to a random swarm of strategy again after a certain point. Figure 6 illustrates these hypothesis.

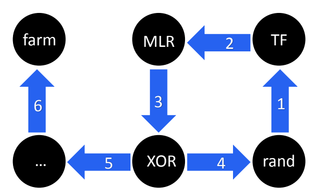


Fig. 6: Illustration for a hypothetical Strategy Invasion Map

Remark In some way you could extrapolate the theory that "invasion" and "increased network complexity" are related since the literature suggests that strategies performance performs better with more complex structures [50]. It is however true that the likelihood of overfitting increases as one adds hidden layers but we have also seen with Shallow Learning that adding hidden layers can also allow us to do regularization which removes the last hurdle argument against DL.

2) A first attempt at formalizing the evolutionary process: With the aim of providing intuition with respect to the sort of interactions that occurs between strategies, we formalize the Evolutionary Process (EP). For this we summarize briefly the recent work we have done around this topic using the same jargon than in our initial papers [37], [38].

Strategy A strategy will consist of an HFFF $\mathcal{H}$, a rolling P&L $\mathcal{P}$ and a common orderbook $\mathcal{O}$ as shown by Equation (13).

$$\mathcal{S} \triangleq \{\mathcal{P}, \mathcal{H}, \mathcal{O}\} \tag{13}$$

[23] I am however personally skeptical on the results of these published studies but I do accept the potential of CNN in trading.

[24] starting with a shallow NN and increasing in complexity in order to understand whether the universal features learnt are because the NN is deep or is it because it has a hidden layer.

The initial research consisted [37], [38] of creating a strategy tournament in which random strategies in HFFF format would be generated. Each random strategy would read the orderbook, follow the instruction of its HFFF weights to put orders on the orderbook. The underlier moves as a result of these orders. Finally a genetic algorithm is applied in which the strategies are ranked according to the P&L with the best strategies reproducing with a small mutation, the average strategies would remain without reproducing and the worst strategies would die. The summary of this process is represented by Figure 7. Following Cedric Villani's [56]

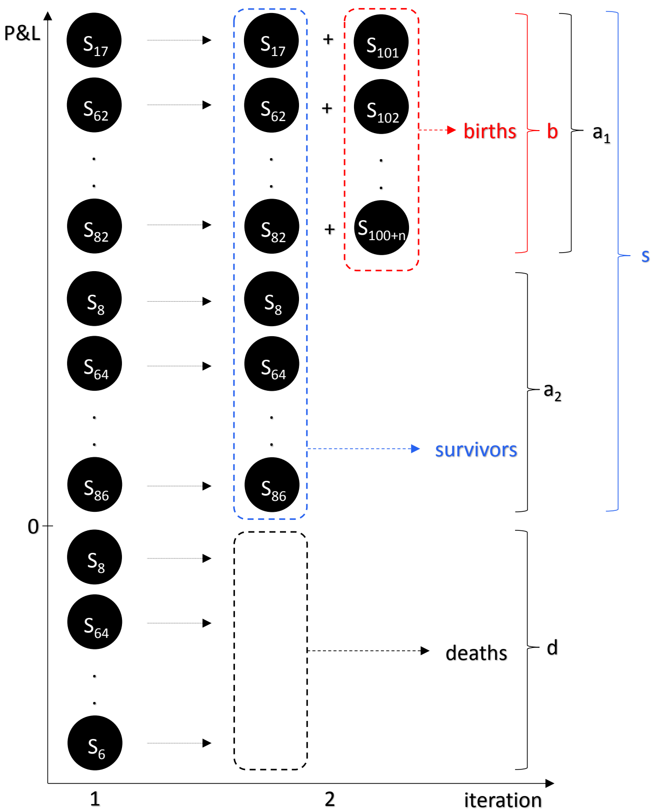


Fig. 7: Death and Birth processes in our GA

comment on the relationship between theory and simulation, more specifically around how simulations can give us good intuition about the theory, we decided to abandon this scientific approach as interpretation proved to be quite difficult [37], [38] mainly but not limited to the issue of strategy classification.

Definition (Strategy Classification): We label N_k the number of total alive strategies, N_k^e the number of tr**e**nd following like strategies, N_k^m the number of **m**ulti-linear regression like strategies, N_k^r the number of xo**r** like strategies and N_k^o the number of ***o**ther unclassified* strategies. The relationship between these entities can be summarized by Equation (14).

$$N_k = N_k^e + N_k^m + N_k^r + N_k^o \tag{14}$$

B. Path of Interaction

Our first few simulations, despite not fulfilling the burden of proof, opened our eyes up to issues associated to optimality, need for more scientific rigour and perhaps an alternative way of fulfilling this burden of proof. The concept of Path of Interaction that we introduce next is an attempt at addressing this alternative methodology.

1) HFTE Game: One way to control our simulation issues, is to perhaps take a step back in complexity in order to gain momentum in constructing a theory with more rigor. With this in mind we have chosen to inspire ourself from the scientific method used by Axelrod [2], [3] extended by Nowak's [46], [49], and to introduce a mathematical object, similar in spirit to the PD matrix used as a battle ground by the name of Path of Interaction. In order to do this rigorously. Let us first go through few definitions.

Definition (Dynamic Mini Order-Book): We will call a Dynamic Mini Order-Book o, the sequence of length l of static snapshots of the order-book ${}^{a_2,a_1}\mathtt{M}_{b_1,b_2}$ of asked and bid volumes a_i/b_i where i corresponds to the depth of the order book and $\mathtt{M}$ its mid price.

Remark In the context of our study we will take $l = 4$.

Definition (Ranking Rule): A Ranking Rule are the set of directives that decides the Birth, Death and Survival processes of any Strategy Ecosystem.

Definition (Environment): We will call an Environment e of size i a set of evolving strategy, $S = s^a, s^b, \ldots, s^i$ of HFFF spanning the one from Figure 3 with potential to interact with each other one after the other via an order-book, ${}^{a_2,a_1}\mathtt{M}_{b_1,b_2}$.

Remark Note that the Ranking Rules we assume going forward are the one described by Figure 7. The environment can then evolves according to a set of Ranking Rules.

Definition (HFTE Game): We will call an HFTE Game the sequence of Environments composed of 2 strategies, $S = s^a, s^b, \ldots, s^i$ of HFFF spanning the one from Figure 3 with a dynamic mini order-book and P&L.

Definition (Full Order-Book (FOB)): An OB will be called full if and only if it has a volume of 1 on all the depth of the OB.

Definition (Path of Interaction Table): We will call a Path of Interaction Table an HFTE Game decomposed in its most infinitesimal steps.

Table II represents an example of the latter definition. The top row of the table points to the strategy involved in the relevant column. The row below (2nd row from the top) provides the stage of the HFTE Game. The 3rd row corresponds to the trading signal. The game starts in a states of in which none of the two strategies has a position (Signal = "N/A") on the order book. Because each strategy needs some form of information on the order book, we take as assumption that there is a random seed on the order book. There is four possibilities of random seeds corresponding to whether the price has been going up or down last and whether the order book has increased its OI or decreased it. These four situations are symbolized by the following set of symbols: $\upuparrows$, $\uparrow\downarrow$, $\downarrow\uparrow$ and $\downdownarrows$. We have chosen the case of $\upuparrows$ to illustrate our examples arbitrarily. The 4th row corresponds to the order book state. The latter can be either scarse or full. We will see that this latter point matters but for now let us illustrate this point with an example. In Table II, we start $\mathtt{P}^{1,1,1,1,1,1}$ meaning that at the current price P, we have one order to sell at the first 6 depths of the order book. The 5th row corresponds to the current price (last completed order) or the mid price if no order was completed in the current iteration. The 6th row corresponds to the Open Interest. If the buy side of the order book has one of its orders matched then the OI decreases by 1 (-1 if the opposite occurs). The 7th row corresponds to the price change. If no order is matched, then the price is approximated by the mid price. The last row corresponds to the profit and loss.

Example In order to illustrate the Path of Interaction we propose to go through the details of a TF strategy interacting with another TF. Algorithm 2 represents our simplified TF strat and Table II represents the Path of Interaction of two strategies following the systematic rules of Algorithm 2. In this table, for convenience sake[25], we have chosen to represent only one side of the order book: $\mathtt{P}^{1,1,1,1,1,1}$ (for display purposes seeing that the price only takes one direction in the simulations). Since both strategies follow the trend, and that the order book is full, the price keeps increasing, their respective P&L keeps increasing and the OI imbalance keeps decreasing. Table II can therefore be seen as a rigorous proof that the TF strategies interacting with each other is "self fulfilling", a terminology we introduce more rigorously next.

Algorithm 2: Simplified TF Strategy

Input: $s, \Delta O, \Delta P$
Output: o
  ▷ Our simple TF Strategy copies last update's trend while disregarding OI
if $\Delta P > 0$ **then**
  order$\leftarrow 1$
else if $\Delta P < 0$ **then**
  order$\leftarrow -1$
else
  order$\leftarrow 0$
return order

2) Invasion Flow Chart: We would like to introduce the concept of Invasion Flow Chart. It can be intuitively understood as being the mirror concept of evolutionary dynamics applied to quantitative strategies through the mean of the HFTE Game instead of the Prisoners Dilemma Matrix.

[25]The price dynamics goes in only one direction in this case.

Definition (Invasion): We will call a strategy, s invasive with respect to an environment, e when the P&L of s increases through the HFTE Games taking place in the environment e.

Example For instance, if we assume that, the more complex a network is, the more likely it is to invade, up to a point where overfitting makes the network obsolete in it performance then we would expect to see an invasion flowchart like the one in Figure 6. Indeed if we assume a TF brings some sort of information innovation from a random strategy and if we assume that the MLR sees more information than the TF and so on then Figure 6 represents a flow chart that exhibits the idea that TF strategies would invade an environment composed of random strategies, that TF would in turn be invaded by MLR, which would be invaded by XORs etc ... This chart also assume that beyond XOR strategies, the complexity would be such that it would equate to a random strategy or would alternatively take a complex path which would lead to a farmer like strategy. We will illustrate later on in this Section that hypothesis illustrated by Figure 6 is not necessarily verified.

Definition (Self-Fulfilling): We will call a strategy, s Self-Fulfilling when it is Invasive with respect to an environment composed of strategies like itself.

3) New Strategy Tournament: Before we discuss our Strategy Tournament, in order to avoid the classification issues mentioned earlier in our research [38], [37], we decided for this new tournament to take three key strategies most simple forms. The first of these three strategies is the simplified TF strategy from Algorithm 2, the second is the simplified MLR strategyThe idea of this simplified version is that Price and OB imbalance both contribute in defining the trading signal. formalized by Algorithm 3 and finally the simplified XOR strategy from Algorithm 4. A Path of Interaction tournament was implemented in the context of 15 possible games on 7 different timescales: $0, 2, 3, 5, 11, 23, 47$. The choice of these timescales may be a little odd at first glance but the idea was to increase the timescale on average by a factor of two while at the same time picking prime numbers. Though, this may sound like unnecessary complexity, the idea of the latter is related to an intuition that we

Algorithm 3: Simplified MLR Strategies

Input: $s, \Delta O, \Delta P$
Output: o
  ▷ Simplified MLR Strategy follows the trend until basic OB imbalance
if $\Delta O + 2 \times \Delta P > 0$ **then**
  order$\leftarrow 1$
else if $\Delta O + 2 \times \Delta P < 0$ **then**
  order$\leftarrow -1$
else
  order$\leftarrow 0$
return order

Strategy	seed $\upuparrows$	TF1	TF2	TF1	TF2	TF1	TF2
Iteration	0		1		2		3
Signal	N/A	+1	+1	+1	+1	+1	+1
OB	$\mathtt{P}^{1,1,1,1,1,1}$	${}_{0}\mathtt{P}^{1,1,1,1,1}$	${}_{0,0}\mathtt{P}^{1,1,1,1}$	${}_{0,0,0}\mathtt{P}^{1,1,1}$	${}_{0,0,0,0}\mathtt{P}^{1,1}$	${}_{0,0,0,0,0}\mathtt{P}^{1}$	${}_{0,0,0,0,0,0}\mathtt{P}$
Mid	100	101	+102	103	104	105	106
ΔOI	$+1$	-1	-2	-3	-4	-5	-6
ΔPrice	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$
P&L	$[0,0]$		$[1,0]$		$[2,1]$		$[3,2]$

TABLE II: Path of Interaction for 2 TF Strategies with $\upuparrows$ Seeds and Full OB

Algorithm 4: Simplified XOR Strategies

Input: $s, \Delta O, \Delta P$
Output: o
▷ Defining simplified XOR Strategy
if $(\Delta O > 0)$ & $(\Delta P > 0)$ **then**
  order← 1
else if $(\Delta O > 0)$ & $(\Delta P < 0)$ **then**
  order← -1
else if $(\Delta O < 0)$ & $(\Delta P > 0)$ **then**
  order← -1
else if $(\Delta O < 0)$ & $(\Delta P < 0)$ **then**
  order← 1
else
  order← 0
return order

had over potential cycles occurring in these games. Though formalizing these possible cycles is potentially premature, we thought of putting them in place preemptively to avoid possible chances of getting into cycles (which would have made analyzing these interactions harder).

Remark In order to use some conventions around strategy sequences for HFTE games we have chosen the following notation $\xhookrightarrow{s_1} s_2$ and $s_2 \xhookrightarrow{s_1} s_3$ to mean, for the first case, that strategy s_1 changes first the OB, then s_2 (and the sequence continues until the end of the timescale) and, for the second case s_3 impacts the OB after s_2 (before, again going back to s_1). For example, $TF \hookrightarrow TF$ means that the environment is composed of two TF strategies and $MLR \xhookrightarrow{TF} XOR$ refers to an HFTE game composed of a TF, MLR and XOR strategy which OB impact sequence is one which mimics the intuitive order laid down by the $\hookrightarrow$ symbol (TF, first, MLR, second and XOR, third). These symbols are expended into their full form in Tables III and IV but we thought it would be useful to have a text friendlier version for the analysis.

Table III represents the results of these games for two strategies interacting and Table IV represents the same for 3 strategies. We can make several interesting observations.

Proposition The TF strategy is self-fulfilling on a OB that is full.

Proof: We have illustrated this point with Table II. Though only on 4 iterations, the proof can be expanded on longer timescales using recursion. □

Remark The intuition we had [38] around the TF acting like a prey increasing exponentially in frequency in the absence of predator is confirmed. The first connections to the Lotka-Volterra 3-species predator/prey model is established. It is worthy to note however that there is a benefit in starting first as the TF1 does better at the end in this HFTE game.

Proposition A strategy A can invade a strategy B but the latter can invade the same strategy B if the seed or and the sequence in which these strategies are started changes.

Example The MLR strategy invades the TF strategy on the longer times scales (column s_2 of Table III) but when the MLR starts the HFTE game (column s_4 of Table III), then the TF strategy invades the MLR strategy. The same remark can be made when the XOR strategy take the MLR spot in the same HFTE set up (column s_3 and s_7 of Table III).

Proposition The Dominance relation is not transitive.

Example This comes to exposing that if a strategy A dominates a Strategy B and Strategy B dominates Strategy C, this does not mean that Strategy A will dominate Strategy C. A counterexample for this point is given by s_2, s_6 and s_3 of Table III.

Proposition Having a more complex strategy does not mean a higher P&L.

Example We can observe in column s_7 of Table III, that the TF strategy invades the XOR strategy over the first 47 iterations even-though the XOR strategy involves a hidden layer, on the contrary to the TF strategy that consist of only 1 input.

Proposition All strategies can make money even if the market goes down.

Example See s_6 example in Table III. □

Proposition Starting first is not always an advantage.

Example See s_5 in Table III for the example (even with twin strategies). □

Scenario	$TF \hookrightarrow TF$	$TF \hookrightarrow MLR$	$TF \hookrightarrow XOR$	$MLR \hookrightarrow TF$	$MLR \hookrightarrow MLR$	$MLR \hookrightarrow XOR$	$XOR \hookrightarrow TF$	$XOR \hookrightarrow MLR$	$XOR \hookrightarrow XOR$
Code	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9
P&L$_0$	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\end{bmatrix}$
ΔP_0	0	0	0	0	0	0	0	0	0
P&L$_2$	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\end{bmatrix}$
ΔP_2	0	0	0	0	0	0	0	0	0
P&L$_3$	$\begin{bmatrix}3\\2\end{bmatrix}$	$\begin{bmatrix}3\\2\end{bmatrix}$	$\begin{bmatrix}-1\\2\end{bmatrix}$	$\begin{bmatrix}3\\2\end{bmatrix}$	$\begin{bmatrix}3\\2\end{bmatrix}$	$\begin{bmatrix}-1\\2\end{bmatrix}$	$\begin{bmatrix}3\\2\end{bmatrix}$	$\begin{bmatrix}3\\2\end{bmatrix}$	$\begin{bmatrix}-1\\2\end{bmatrix}$
ΔP_3	3	3	-1	3	3	-1	3	3	-1
P&L$_5$	$\begin{bmatrix}9\\7\end{bmatrix}$	$\begin{bmatrix}-9\\8\end{bmatrix}$	$\begin{bmatrix}-1\\4\end{bmatrix}$	$\begin{bmatrix}-1\\10\end{bmatrix}$	$\begin{bmatrix}9\\5\end{bmatrix}$	$\begin{bmatrix}4\\5\end{bmatrix}$	$\begin{bmatrix}3\\5\end{bmatrix}$	$\begin{bmatrix}3\\9\end{bmatrix}$	$\begin{bmatrix}7\\-2\end{bmatrix}$
ΔP_5	6	-3	-3	5	-5	-4	5	-4	3
P&L$_{11}$	$\begin{bmatrix}45\\40\end{bmatrix}$	$\begin{bmatrix}-39\\21\end{bmatrix}$	$\begin{bmatrix}-1\\4\end{bmatrix}$	$\begin{bmatrix}-50\\78\end{bmatrix}$	$\begin{bmatrix}15\\12\end{bmatrix}$	$\begin{bmatrix}14\\7\end{bmatrix}$	$\begin{bmatrix}-9\\26\end{bmatrix}$	$\begin{bmatrix}-16\\21\end{bmatrix}$	$\begin{bmatrix}7\\-2\end{bmatrix}$
ΔP_{11}	15	11	-3	13	15	-6	11	11	3
P&L$_{23}$	$\begin{bmatrix}198\\187\end{bmatrix}$	$\begin{bmatrix}-216\\48\end{bmatrix}$	$\begin{bmatrix}-1\\4\end{bmatrix}$	$\begin{bmatrix}-326\\387\end{bmatrix}$	$\begin{bmatrix}39\\38\end{bmatrix}$	$\begin{bmatrix}74\\11\end{bmatrix}$	$\begin{bmatrix}-87\\122\end{bmatrix}$	$\begin{bmatrix}-21\\54\end{bmatrix}$	$\begin{bmatrix}7\\-2\end{bmatrix}$
ΔP_{23}	33	26	-3	28	36	-10	23	27	3
P&L$_{47}$	$\begin{bmatrix}828\\805\end{bmatrix}$	$\begin{bmatrix}-703\\354\end{bmatrix}$	$\begin{bmatrix}-1\\4\end{bmatrix}$	$\begin{bmatrix}-1580\\1707\end{bmatrix}$	$\begin{bmatrix}96\\99\end{bmatrix}$	$\begin{bmatrix}290\\19\end{bmatrix}$	$\begin{bmatrix}-459\\530\end{bmatrix}$	$\begin{bmatrix}-27\\54\end{bmatrix}$	$\begin{bmatrix}7\\-2\end{bmatrix}$
ΔP_{47}	69	-75	-3	58	78	-18	47	27	3

TABLE III: P&L in Path of Interaction for 2 Strategies with $\upuparrows$ Seeds and Full OB

Scenario	$MLR \xhookrightarrow{TF} XOR$	$XOR \xhookrightarrow{TF} MLR$	$TF \xhookrightarrow{MLR} XOR$	$XOR \xhookrightarrow{MLR} TF$	$TF \xhookrightarrow{XOR} MLR$	$MLR \xhookrightarrow{XOR} TF$
Code	s_{10}	s_{11}	s_{12}	s_{13}	s_{14}	s_{15}
P&L$_0$	$\begin{bmatrix}0\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\0\end{bmatrix}$
ΔP_0	0	0	0	0	0	0
P&L$_2$	$\begin{bmatrix}0\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\0\end{bmatrix}$
ΔP_2	0	0	0	0	0	0
P&L$_3$	$\begin{bmatrix}3\\2\\0\end{bmatrix}$	$\begin{bmatrix}-2\\3\\3\end{bmatrix}$	$\begin{bmatrix}3\\2\\0\end{bmatrix}$	$\begin{bmatrix}-1\\2\\0\end{bmatrix}$	$\begin{bmatrix}-1\\-2\\2\end{bmatrix}$	$\begin{bmatrix}3\\2\\0\end{bmatrix}$
ΔP_3	3	-2	3	-1	-1	3
P&L$_5$	$\begin{bmatrix}4\\0\\4\end{bmatrix}$	$\begin{bmatrix}-2\\-8\\0\end{bmatrix}$	$\begin{bmatrix}8\\-5\\7\end{bmatrix}$	$\begin{bmatrix}5\\6\\4\end{bmatrix}$	$\begin{bmatrix}-11\\-3\\20\end{bmatrix}$	$\begin{bmatrix}-3\\7\\-4\end{bmatrix}$
ΔP_5	4	3	-4	-5	-6	-3
P&L$_{11}$	$\begin{bmatrix}-26\\60\\-21\end{bmatrix}$	$\begin{bmatrix}-108\\61\\-36\end{bmatrix}$	$\begin{bmatrix}17\\-7\\-34\end{bmatrix}$	$\begin{bmatrix}14\\-18\\-37\end{bmatrix}$	$\begin{bmatrix}10\\-26\\25\end{bmatrix}$	$\begin{bmatrix}-48\\57\\-47\end{bmatrix}$
ΔP_{11}	17	-14	7	6	18	16
P&L$_{23}$	$\begin{bmatrix}-65\\-13\\119\end{bmatrix}$	$\begin{bmatrix}-164\\131\\-39\end{bmatrix}$	$\begin{bmatrix}57\\-35\\-198\end{bmatrix}$	$\begin{bmatrix}54\\-62\\-201\end{bmatrix}$	$\begin{bmatrix}128\\-93\\-74\end{bmatrix}$	$\begin{bmatrix}137\\145\\-96\end{bmatrix}$
ΔP_{23}	31	-32	15	14	45	-46
P&L$_{47}$	$\begin{bmatrix}-3127\\2500\\-231\end{bmatrix}$	$\begin{bmatrix}-720\\250\\-289\end{bmatrix}$	$\begin{bmatrix}233\\-187\\-910\end{bmatrix}$	$\begin{bmatrix}230\\-246\\-913\end{bmatrix}$	$\begin{bmatrix}187\\-230\\-588\end{bmatrix}$	$\begin{bmatrix}553\\621\\13\end{bmatrix}$
ΔP_{47}	-54	-54	31	30	97	-104

TABLE IV: P&L in Path of Interaction for 3 Strategies with $\upuparrows$ Seeds and Full OB

4) Few Interesting Hypotheses: Finally we wanted to end this part by suggesting few hypotheses based on some of our observations.

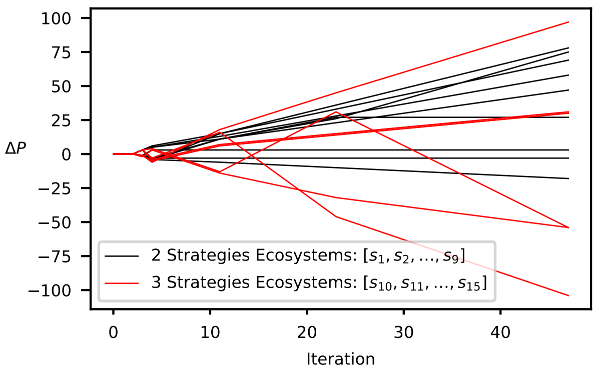


Fig. 8: Instability increases with an additional strategy.

Hypothesis: In the finite horizon we have looked at, we noticed that all strategies in our ecosystems can increase their P&L at the same time but all cannot decrease their P&L at the same time. This observation, again on a finite horizon may not be true as $t \to \infty$.

We noticed this interesting fact with our relatively small sample of HFTE games but have not been able to find a counter example yet nor been able to rigorously prove it. The proof might be easier than it seems, using perhaps the pigeonhole principle but we have not been able to formalize the proof or a sketch.

Hypothesis: Similar physical laws drive *morality*[26] and HFTE games.

We explain next this seemingly odd and unexpected terminology. In the spirit of using simple rules at the agent level as a triggering point to complex interactions in the ecosystem that can turn into laws, we thought that this second hypothesis would also be inspiring. This latter proposed hypothesis might not be immediately obvious but there seems to be interesting connection between the TF strategy in an HFTE game and the TFT strategy in Axelrod's [2], [3] computer tournament described our previous paper [38]. As a reminder the TFT strategy cooperates first and continues doing so until it is deceited, upon which it deceits on the next move. However, the TFT has the ability to forgive. This means that, if the opponent agent decides to cooperate again then the TFT, starts cooperating on the next encounter. The TFT is therefore considered a nice strategy [15] but adaptable at the same time [46]. So how does that relate to the TF in finance? Both are successful strategies yet are very simple. They both replicate the last agent's move: so they are both "cooperative"[27] but are adaptable[28].

Remark This comparison may not seem quite apropos at first or at least it may not be intuitive. This may be related to the negative bias we have against the moral aspects of Finance. These are due to many elements but one contributing factor is due to some of the misconducts in HFT which are more related to unfair advantages in technology or immoral actions [13], [43] taken in an unfair game based on asymmetric information about the market. In our research technological advantages are not taken into consideration.

Hypothesis: Diversity in financial strategies in the market lead to its instability.

Remark Finally we proposed a previously introduced hypothesis we wanted to raise before concluding this part: we noticed, in Figure 8 that the 3 strategies ecosystems exhibited more fluctuations than the 2 strategies ecosystems which tend to support the hypothesis than more diversity in an ecosystem of strategies induces more instability[29] to the market. It does however suggests it empirically but this example does not constitute obviously a proof. Also, a bigger ecosystem allows for more time for a random walk to depart from its expected mean so to some extent the fact that the 3 strategies ecosystem has a higher variance is explained by that increased timescale but the increased fluctuation seems to go beyond what is expected by the addition of an additional step. It is also important to point that the stability and diversity debate has had an interesting breakthrough recently [4]. More specifically, the stability of the equilibria reached by ecosystems formed by a large number of species with strong and heterogeneous interactions (therefore more realistic ecological niches) the system displays multiple equilibria which are all marginally stable. Though this studies is applied to biological niches, it is not difficult to imagine that a similar result could be found in an algorithmic trading ecosystem. This would therefore contradict the hypothesis we have put forward.

In this part we have built the humble start of a schemes involving the Bottom-Up approach to algorithmic trading. We first attempted to reach that objective by tackling the problem using a simple genetic algorithm methodology. Though intuitive an interesting, we abandoned this approach because of a series of problem associated, but not limited to, classification, lack of visibility and lack of optimality. We however, took this opportunity to shown possible connections to other STEM fields and how they could be brought in the world of QF through the regulatory door. To study the problem with more visibility, rigour, and in order to gain momentum, we took a step back in the scientific approach and formalized the HFTE game as well as the Path of

[26] We refer here to some of the work associate the formal mathematical definition of morality [15], more specifically in the context of cellular automaton and the iterative prisoners dilemma [3], [2], [46].

[27] Cooperative in evolutionary dynamics seem to translate into "trending" in QF.

[28] The TF can change his position on the market if the trend changes.

[29] This assertion could be challenged with a simulation which additional strategies (such as potentially a market making strategy) would be designed in order to create this stability [9].

Interaction concepts. We have also given 15 different kinds of HFTE games split on 7 different timescales and also presented few interesting observations about the interesting complexity in the relationship of these strategies, even when simplified. This study was done with the premise that we knew what strategies were involved in the ecosystem and in which sequence they act upon this ecosystem. Though simplistic, in the choice of the available strategies[30], the current model give a good overview of how the method could be enhanced by simply adding more strategies including market makers. However, market participants are quite secretive in reality when it comes to their financial strategies. The only observable data on the market is essentially the price dynamics and the order book. We explore in the next chapter how inference can be constructed in the Bottom-Up approach when the price dynamic alone is available.

IV. Building a Realistic Market Simulator through Multi-Target Tracking

In this section we first go over a literature review of the Multi-Target Tracking. We then expand the study by connecting some of the concepts in the previous two Sections with particle filtering applied to scenario modelling and finally show how we can use this methodology to construct a realistic market simulators in which strategies can be tested.

A. Particle Filter Methodology

1) Importance Sampling: Importance sampling (IS) was first introduced in [41] and but then expanded [24]. The objective of importance sampling is to sample the distribution in the region of importance in order to achieve computational efficiency via lowering the variance. More specifically we would like to choose a proposal distribution $q(x)$ in place of the true, harder to sample probability distribution $p(x)$. The main constraint of this method is to make sure the support of $q(x)$ covers that of $p(x)$. In Equation (15a) we write the integration problem in the more appropriate form with Equation (15b) the numerical approximation where N_p, usually describes the number of independent samples drawn from $q(x)$ to obtain a weighted sum to approximate $\hat{f}$.

$$\int f(x)p(x)dx = \int f(x)\frac{p(x)}{q(x)}q(x)dx \tag{15a}$$

$$\hat{f} = \frac{1}{N_p}\sum_{i=1}^{N_p} W(x^{(i)})f(x^{(i)}) \tag{15b}$$

In Equation (16a) $W(x^{(i)})$ is the Radon-Nikodym derivative of $p(x)$ with respect to $q(x)$ or called in engineering the importance weights. Equation (16b) suggests that if the normalizing factor for $p(x)$ is not known, the importance weights can only be evaluated up to a normalizing constant.

$$W(x^{(i)}) = \frac{p(x^{(i)})}{q(x^{(i)})} \tag{16a}$$

$$W(x^{(i)}) \propto p(x^{(i)})q(x^{(i)}) \tag{16b}$$

[30]The presence of market makers would make the results more interesting [9].

To ensure that $\sum_{i=1}^{N_p} W(x^{(i)}) = 1$, we normalize the importance weights to obtain Equation (17).

$$\hat{f} = \frac{\frac{1}{N_p}\sum_{i=1}^{N_p} W(x^{(i)})f(x^{(i)})}{\frac{1}{N_p}\sum_{i=1}^{N_p} W(x^{(i)})} = \frac{1}{N_p}\sum_{i=1}^{N_p} \tilde{W}(x^{(i)})f(x^{(i)}) \tag{17}$$

where $\tilde{W}(x^{(i)}) = \frac{W(x^{(i)})}{\sum_{i=1}^{N_p} W(x^{(i)})}$ are called the normalized importance weights.

2) Sequential Monte Carlo Methods: Sequential Monte Carlo methods (SMC), also known as Particle Filters (PF) are statistical model estimation techniques based on simulation. They are the sequential (or 'on-line') analogue of Markov Chain Monte Carlo (MCMC) methods and similar to importance sampling methods. If they are elegantly designed they can be much faster than MCMC. Because of their non linear quality they are often an alternative to the Extended Kalman Filter (EKF) or Unscented Kalman Filter (UKF). They however have the advantage of being able to approach the Bayesian optimal estimate with sufficient samples. They are technically more accurate than the EKF or UKF. The aims of the PF is to estimate the sequence of hidden parameters, x_k for $k = 1, 2, 3, \ldots$, based on the observations y_k. The estimates of x_k are done via the posterior distribution $p(x_k|y_1, y_2, \ldots, y_k)$. PF do not care about the full posterior $p(x_1, x_2, \ldots, x_k|y_1, y_2, \ldots, y_k)$ like it is the case for the MCMC or importance sampling (IS) approach. Let's assume x_k and the observations y_k can be modeled in the following way: $x_k|x_{k-1} \sim p_{x_k|x_{k-1}}(x|x_{k-1})$ and with given initial distribution $p(x_1)$, $y_k|x_k \sim p_{y|x}(y|x_k)$. Equations (18a) and (18b) gives an example of such system.

$$x_k = f(x_{k-1}) + w_k \tag{18a}$$

$$y_k = h(x_k) + v_k \tag{18b}$$

It is also assumed that $cov(w_k, v_k) = 0$ or w_k and v_k mutually independent and iid with known probability density functions. $f(\cdot)$ and $h(\cdot)$ are also assumed known functions. Equations (18a) and (18b) are our state space equations. If we define $f(\cdot)$ and $h(\cdot)$ as linear functions, with w_k and v_k both Gaussian, the KF is the best tool to find the exact sought distribution. If $f(\cdot)$ and $h(\cdot)$ are non linear then the Kalman filter (KF) is an approximation. PF are also approximations, but convergence can be improved with additional particles. PF methods generate a set of samples that approximate the filtering distribution $p(x_k|y_1, \ldots, y_k)$. If N_P in the number of samples, expectations under the probability measure are approximated by Equation (19).

$$\int f(x_k)p(x_k|y_1, \ldots, y_k)dx_k \approx \frac{1}{N_P}\sum_{L=1}^{N_P} f(x_k^{(L)}) \tag{19}$$

Sampling Importance Resampling (SIR) is the most commonly used PF algorithm, which approximates the probability measure $p(x_k|y_1, \ldots, y_k)$ via a weighted set of N_P particles $(w_k^{(L)}, x_k^{(L)})\ :\ L = 1, \ldots, N_P$. The importance weights $w_k^{(L)}$ are approximations to the relative posterior probability measure of the particles such that $\sum_{L=1}^{P} w_k^{(L)} =$

Algorithm 5: Sequential Monte Carlo

Input: array of weights w_p^N, $\pi(x_k|x_{1:k-1}^{(L)}, y_{1:k})$
Output: array of weights w_p^N resampled
Sample:
for $L = 1$ *to* N_P **do**
  $x_k^{(L)} \sim \pi(x_k|x_{1:k-1}^{(L)}, y_{1:k})$
for $L = 1$ *to* N_P **do**
  $\hat{w}_k^{(L)} \leftarrow w_{k-1}^{(L)} \frac{p(y_k|x_k^{(L)})p(x_k^{(L)}|x_{k-1}^{(L)})}{\pi(x_k^{(L)}|x_{1:k-1}^{(L)}, y_{1:k})}$
for $L = 1$ *to* N_P **do**
  $w_k^{(L)} \leftarrow \frac{\hat{w}_k^{(L)}}{\sum_{J=1}^{P} \hat{w}_k^{(J)}}$
$\hat{N}_{eff} \leftarrow \frac{1}{\sum_{L=1}^{P} \left(w_k^{(L)}\right)^2}$
Reample:
draw N_P particles from the current particle set with probabilities proportional to their weights. Replace the current particle set with this new one.
if $\hat{N}_{eff} < N_{thr}$ **then**
  for $L = 1$ *to* N_P **do**
    $w_k^{(L)} \leftarrow 1/N_P$.

1. SIR is a essentially a recursive version of importance sampling. Like in IS, the expectation of a function $f(\cdot)$ can be approximated like described in Equation (20).

$$\int f(x_k)p(x_k|y_1, \ldots, y_k)dx_k \approx \sum_{L=1}^{N_P} w^{(L)} f(x_k^{(L)}) \qquad (20)$$

The algorithm performance is dependent on the choice of the proposal $\pi(x_k|x_{1:k-1}, y_{1:k})$ distribution with the optimal proposal distribution being $\pi(x_k|x_{0:k-1}, y_{0:k})$ in Equation (21).

$$\pi(x_k|x_{1:k-1}, y_{1:k}) = p(x_k|x_{k-1}, y_k) \qquad (21)$$

Because it is easier to draw samples and update the weight calculations the transition prior is often used as importance function: $\pi(x_k|x_{1:k-1}, y_{1:k}) = p(x_k|x_{k-1})$. The technique of using transition prior as importance function is commonly known as Bootstrap Filter and Condensation Algorithm. Figure 9 gives an illustration of the algorithm just described and Algorithm (5) summarizes the SMC methodology.

3) Resampling Methods: Resampling methods are usually used to avoid the problem of weight degeneracy. More specifically avoiding situations where our trained probability measure tends towards the Dirac distribution must be avoided because it really does not give much information on all the possibilities of our state. There exists many different resampling methods, Rejection Sampling, Sampling-Importance Resampling, Multinomial Resampling, Residual Resampling, Stratified Sampling, and the performance of our algorithm can be affected by the choice of our resampling method. The stratified resampling proposed by Kitagawa [30] is optimal in terms of variance. Figure 9 gives an illustration of the latter and the corresponding algorithm is described in algorithm 6. We see at the top of the Figure 9 the discrepancy between the estimated pdf at time t and the real pdf. Random numbers from $[0, 1]$ are drawn subsequently, and depending on the importance of these particles they are moved to more useful places so as to gain proximity with the real PDF.

Algorithm 6: Resample

Input: array of weights w_1^M
Output: array of weights w_1^M resampled

Sample:
$u^0 \sim \mathcal{U}[0, 1/M]$

Resample:
for $m = 1$ *to* N **do**
  $i^{(m)} \leftarrow \left\lfloor (w_n^{(m)} - u^{(m-1)}m) \right\rfloor + 1$
  $u^{(m)} \leftarrow u^{(m)} + \frac{i^{(m)}}{M} - w_n^{(m)}$

B. Scenario Tracking Algorithm

1) A Brief Introduction: Recently, SMC methods [17], [18], [34], especially when it comes to the data association issue, have been developed. An alternative wording for SMC is Particle Filters (PF) [22], [29] and Multi Target Tracking (MTT) for data association. Their popularity is mainly due to their performancefor nonlinear and non-Gaussian problemsContrasting with classic linear methods like the KF/EKF [26].. When applied to our problem, we try to track the ecosystem of strategies through time. Namely we attempt a tracking our state space θ summarized by Equation (22).

$$\theta \triangleq \left\{ \; N_t^s, N_t^b, N_t^d, \cup_{i=1}^{N_t^a} S_i, \cup_{i=1}^{N_t^a} \mathcal{H}_i, \cup_{i=1}^{N_t^a} \mathcal{P}_i, \mathcal{O} \; \right\} \qquad (22)$$

with N_t^s, the number of survived strategies, N_t^b, the number of born strategies, N_t^d, the number of dead strategies and N_t^a, the number of alive strategies[31]. As we can see from Equation (22), not only we need to keep track of the alive strategies through time but also of their HFFF. You also need to keep track of all possible sets of orderbooks $\mathcal{O}$ and P&L $\mathcal{P}$. Though challenging, using a simplified approach can allow us to built complexity in an incremental manner. We discuss this simplified design next leveraging on the study we have found in section III.

2) Simplified Simulation: For this simplified method we assume the state space is limited to a set of 15 scenarios spanned by up to 3 different types of strategies[32] acting on the OB in a sequence that is unknown. In order to manage complexity we have also assumed that there is no birth or death processes involved in our scenarios. Algorithm (7) describes our simplified study in pseudo code.

[31] $N_t^a = N_t^s + N_t^b$ or $N_t^a = N_{t-1}^a + N_t^b - N_t^d$.
[32] Exact formalization has been given by Algorithms 2, 3 and 4.

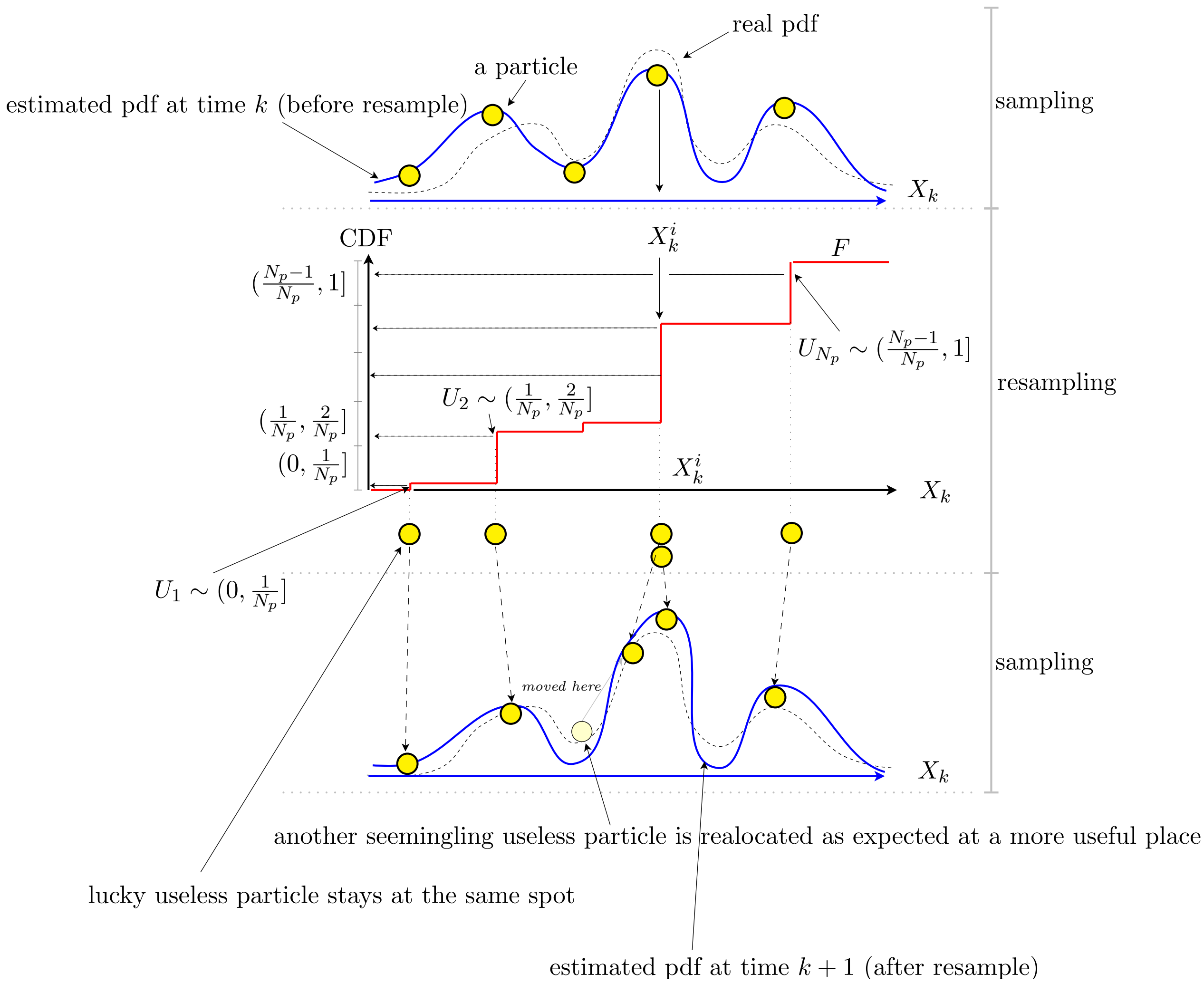


Fig. 9: Stratified Sampling illustration

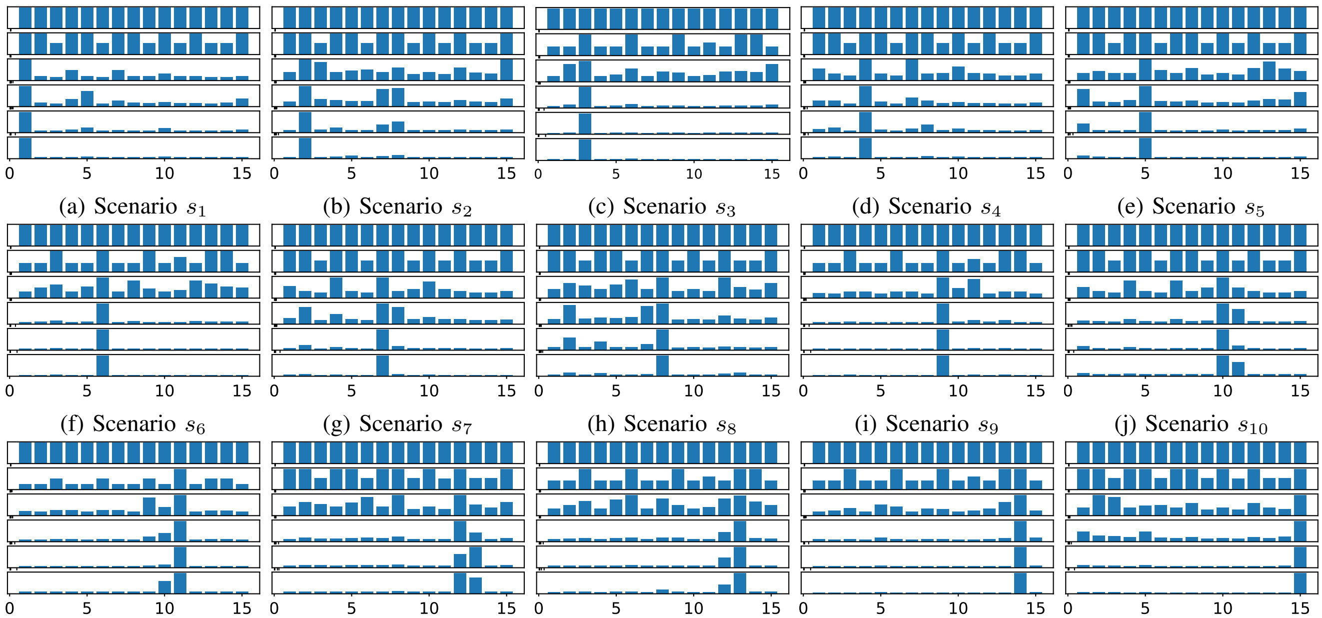
Algorithm 7: Particle Filter Simplified HFTE Strategies

Input: ΔP, I, w_t, λ_e, λ_r
Output: w_t
$w_{t-1} \leftarrow w_t$
$W \leftarrow 0$
for $0 \leqslant s \leqslant 15$ **do**
  ▷ H_I^s is is our Scenario Hash Table for scenario s and iteration I
  $L_s \leftarrow \exp(-\Delta P - H_I^s)$
  $W = W + L_s$
for $0 \leqslant s \leqslant 15$ **do**
  $w_t^s \leftarrow \lambda_e \times \frac{L_s}{W} + (1 - \lambda_e - \lambda_r) \times W_{t-1}^s + \lambda_r \times 1/15$
return w_t

Algorithm 8: Scenarios Hash Table H_I^s

Input: I
Output: array position
if $I == 0$ **then**
  return 0
else if $I == 2$ **then**
  return 1
else if $I == 3$ **then**
  return 2
else if $I == 5$ **then**
  return 3
else if $I == 11$ **then**
  return 4
else if $I == 23$ **then**
  return 5
else if $I == 47$ **then**
  return 6
else
  return 'issue with iteration recognition'

Remark Note that the traditional resampling algorithm, as developed by Doucet [16], has been substituted by the term $W_{t-1}^s + \lambda_r$ in the line $w_t^s \rightarrow \lambda_e \times \frac{L_s}{W} + (1 - \lambda_e - \lambda_r) \times W_{t-1}^s + \lambda_r \times 1/15$. The number of particles are relocated as per a mixture of two distribution. The first of these two



(a) Scenario s_1 (b) Scenario s_2 (c) Scenario s_3 (d) Scenario s_4 (e) Scenario s_5

(f) Scenario s_6 (g) Scenario s_7 (h) Scenario s_8 (i) Scenario s_9 (j) Scenario s_{10}

(k) Scenario s_{11} (l) Scenario s_{12} (m) Scenario s_{13} (n) Scenario s_{14} (o) Scenario s_{15}

Fig. 10: Particle Filter on market scenarios on $[2, 3, 5, 11, 23, 47]$ milestones

being the Gaussian distribution which aim is to engineer the proximity. The second distribution is uniform so as to still allow a minimum number of particles allocated to the other scenarios so as to avoid weight degeneracy.

For this simplified methodology we assumed that the full OB is unknown. The only visible (indirect) information is the price process. The results from the series of simulations are presented partially in Figure 10 with more simulation available in our original paper [37]. What we can observe is that every scenario had already clearly emerged by iteration 23, the second row from the bottom on all 15 scenarios (only 5 are shown in Figure 10). By iteration 47, the density is very clear, so much so that the only reason it is not a Dirac function is due to the resampling methodology introduced in that effect.

C. Building a Realistic Market Simulator

Now that we have shown how to build and detect the market details from the Bottom-Up, how can use our findings to build a realistic market simulator?

1) Context: The context is the following: you are working in a bank, asset manager or in a hedge fund as a "Quant" and have been given the instructions to build a realistic market simulators on which one can test strategies and perhaps decipher market impact or regime change[33]. You are given a set of strategies $\Omega = \{S_1, S_2, \ldots, S_n\}$ that can be replicated through the HFFF (from Figure 3). You also assume that you have a history of P&L for each element of Ω. This simulated market ought to be composed of an ecosystem of all possible theoretical strategies which frequency is unknown and which should react in such a way that the P&L of all strategies for which we have historical data ought to perform in a similar manner. Though, this seems to be an impossible task, there are ways to tackle this problem leveraging on what we have learnt in the previous sections.

2) Proposed Solution: We need to define a particle filter on the scenarios described in the problem formulation. In doing so we need to both define a slightly different likelihood function as well as a very different resampling solution. We need to create a likelihood function for the particles associated to the scenario being investigated. This likelihood function should be itself function of the relative entropy between the expected P&L distribution and the one realized by the simulated market. The Kullback-Leibler divergence [32], of Equation (23) can be a simple enough measure for the individual strategies being simulated. More specifically for discrete probability distributions where H, is the historical distribution, and S, the simulated distribution. The Kullback-Leibler divergence from S to H is given by Equation (23).

$$D_{\mathrm{KL}}(H \| S) = -\sum_i H(i) \log \frac{S(i)}{H(i)}, \tag{23}$$

We can then define the un-normalized likelihood of a market scenario i by Equation (24a) where N_k represents the number of strategies alive as initially defined by Equation (14).

$$L^{(i)} = \sum_{i=1}^{N_k} D_{\mathrm{KL}}(H \| S) \tag{24a}$$

$$\tilde{L}^{(i)} = \frac{L^{(i)}}{\sum_{i=1}^{N_k} L^{(i)}} \tag{24b}$$

Note here the resampling is at first glance done very differently. Indeed we are no longer looking to directly find

[33]Regime change is a fancy expression that essentially means that your strategy is no longer working because the market ecosystem has changed.

the most profitable strategies[34] but the right frequency of each strategy in the ecosystem so as to make the latter more realistic (as compared measure of the historical P&L distribution).

Remark A good argument could be made that selecting for the best strategies using the genetic algorithm resampling methodology of our original paper [39] summarized by figure 7 would converge to a mature market. The latter would hypothetically converge to the ecosystem that would best accommodate the simulation of the historical P&Ls of each strategy. This is not necessarily a bad approach but a little restrictive. An additional way to adjust the resampling here would perhaps be to inspire ourselves from what would be a more rigorous invasion chart of the one from figure 6. The resampling methodology would then consist of increasing/decreasing the neighboring strategies that impact the P&L of the strategies which Kullback-Leibler divergence are the closest to 1 (most off their historical P&L).

V. Conclusion

A. Summary

We have started this paper by pointing to a puzzling observation from the newly born high frequency commodities market which because of its extreme youth and therefore immaturity makes it a great case study for a high frequency market at inception and therefore for our purpose. More specifically as we have seen with Figure 1, fascinating patterned oscillations occurred in the commodities market. These oscillations cannot be explained by the TD assumption in Quantitative Finance. We have proposed to study these oscillations with the BU approach instead, inline with the recommendations of the highest authorities in finance. More specifically this paper was a response to the call for a modelling revolution [5] to occur post subprime crisis in the form of agent based models. The latter theory was developed in 3 Sections. We first expressed classic Financial Strategies in HFFF format and showed the pletora of classic strategies that can be modelled with the HFFF format. We also tried to give the incentive for going from a simple perceptron, to shallow and finally deep learning. We also established possible connections to fields that are traditionally associated to mathematical biology, namely predator-prey models and evolutionary dynamics. This was done in order to expand the mathematical weaponry that we believe have value in 21st century Quantitative Finance. These helped us express the bottom-up approach at the infinitesimal level. More specifically we developed the concept of Path of Interaction in an HFTE Game and proposed 3 hypothesises as a mean to inspire future researchers. In Section IV we looked at how the financial market composition could be tracked through time with MTT. Finally, we used these findings in order to present some guidelines around constructing a market simulator to have more realistic backtests as well as measure market impact without incurring costs.

[34]Or rather, not just yet.

B. Current & Future Research

Our first few simulations opened our eyes up to issues associated to optimality and the need for more scientific rigor. We have classified these points of improvement in half a dozen issues listed below.

1) Classification Simplification: As mentioned before the direct simulation approach [38] is too challenging and the results perhaps too convoluted to filter out the essence of the paper. For this reason we proposed to study the problem using fixed HFFFs of different depth (XOR vs MLR) and width (TF vs MLR). Though this simplifies the problem it also means there is human intervention in the strategy pool chosen. This latter intervention, though convenient raises the question of whether what seems to be equivalent strategies are equivalent after all. Less human interventions should take place going forward.

2) The State Space can be improved: Choosing three types of strategies greatly limits our state space which makes our tracking methodology easier but not as realistic as we wish ultimately. Additional strategies must be incorporated and more HFTE games must be included in our database of scenarios. This could be the work of many years and could be addressed in the form of creating an online database in which interested scientists could deposit their findings in object oriented format for simulation purposes. Generally speaking we need to incorporate a Birth and Death Process to our MTT to make more realistic scenarios. In order to do that we need to incorporate the OB in the likelihood function instead of using only the price dynamics. This will undoubtedly make the programming exercise more challenging but will at the same time bring more value to the research in the long run.

3) Order-book Dynamics: Many of the markets are driven by different rules for the OB. We need to incorporate these different rules in our HFTE games as the latter rules obviously impact the outcome of the games.

4) Increased HFFF complexity does not equate to Invasion: It has been speculated that the need for a bigger brain in humans is partly due to the need for humans to elaborate deceitful strategies with their rivals and cooperative strategies with their allies. It is therefore not entirely ridiculous to associate increased neural network branching (to be roughly understood as increase in cranial size) with increased strategy complexity. However, increased intelligence does not necessarily equate to survival as we can see in the shark population, considered like an apex predator in the sea (but with a relatively small brain), has not evolved for millennial. We are very much at the early stages of defining NN complexity and dominance. A clear picture did not necessarily emerge from the first simulations though an interesting comparison can be made with Axelrod computer tournament [3]. Indeed, Axelrod [3], [37] showed that it was not necessarily the most complicated strategies that prevailed at the end[35]. The TF strategy shares some aspects

[35]Please see TFT strategy in the bigger version of this paper [37] or in the relevant literature [3], [46]: it is a strategy that is simple in the way it adapts to the other strategies.

of the Tit For Tat (TFT) strategy in the sense that they are both simple and adaptive. However, taking the argument in reverse ("complexity pays off" instead of "simple adaptable strategies are best"), can we think about a farming strategy? By this we mean can we come up with a strategy (in a DNN format) that would understand the state of the ecosystem and would take actions based on that ecosystem, deliberately avoiding acquiring alpha on the ecosystem if it felt that it would be beneficial for the long term health of the financial ecosystem? These are fascinating questions that we may figure out sooner than expected.

5) Complex Food Webs: We have seen in Section III-B.1 that we have taken $l = 4$ in our Path of Interaction sequence. Would the Path of Interaction results change if we increase the sequence's length? The answer would be yes if the OB is not full. But what if it remains full? In the context of the Path of Interaction study, is there a more rigorous way to connect some of the Lotka-Volterra predator prey models to these interactions? It seems intuitively more likely that the strategy ecosystem should rather be a complex food web. Can we enhance the idea of the simple Lotka-Volterra predator prey model [57], [12], [37] to more complex food webs? More specifically what are the strategies that would create a stable and unstable food web? The concept of Path of Interaction is meant to be a bridge connecting the gap between strategy formalization to evolutionary dynamics but this bridge in not entirely specified yet.

6) Diversity & Stability: One other legitimate question that we can ask ourselves is whether the HFFF is complex enough to model all financial strategies? And if not all, does it encompass enough strategies to convey something interesting and meaningful when you make the strategies interact with each other. In this context our first paper [38] ended with the proposed "Diversity & the Financial Markets" hypothesis below which is currently an open problem that is interesting to mention in the context of future research:

Diversity & the Financial Markets Diversity in financial strategies in the market leads to its instability.

Remark Note this hypothesis has been studied partially with simulations and can be perhaps indirectly studied or at least intuitively using some of the finding in mathematical biology. More specifically the one associated with diversity in ecosystem and stability[36] [33], [12].

7) Profitability Ecosystem Asymmetry: The second hypothesis that we introduced is as follows:

Ecosystem Profitability Asymmetry hypothesis All strategies in an ecosystem can make money at the same time but all cannot lose money at the same time.

As we mentioned earlier we noticed this interesting fact with our relatively small sample of HFTE games but have not been able to found a counter example yet nor been able to rigorously prove it. It would be relatively easy to incorporate more simulations involving more strategies to see if we can find a counter example. Alternatively, if the hypothesis can be proven then we recommend using the pigeonhole principle for the proof.

8) Morality & HFTE Games: The last hypothesis we introduced is as follows:

Morality & HFTE games hypothesis Similar physical laws drive morality and HFTE games.

The TF strategy in an HFTE game and the TFT strategy in Axelrod's [2], [38] computer tournament seem to have interesting similarities [37] even though they are induced by very different applications of cellular automatons. This suggests that similar physical laws may drive their success though the exact link was not establish. However, we can notice that they are both, simple, cooperative, adaptive and most of the time successful. The relationship seems intuitively quite awkward at first glance. Indeed, Morality and Finance are, at first glance, potentially discordant concepts but the similarities are interesting and certainly worth dwelving more into.

[36] Though no consensus is reached there either.

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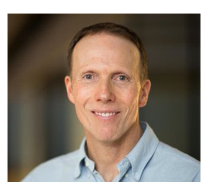

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