# Introducing the **I**mplied **V**olatility surface **P**arameterization (IVP): Application to Options Statistical Arbitrage & Risk Management

Babak Mahdavi Damghani

*bmd@cantab.net*

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# Some of the IVP's Model Performance Criteria

1. **Asset Class Polyvalent**: Should work with different asset classes (moneyness, log-moneyness, delta space ...).
2. **Function Polyvalent**: Trading, Pricing, Compliance & Risk should find it useful.
3. **Dimensionality**: for reasons of computational speed and robustness, the number of **factors** should be **limited but not limiting**.
4. Should be able to **detect and handle arbitrages** within both a trading and risk management context.
5. Inline with today's challenging regulatory environment: **FRTB** friendly, therefore a **flexible liquidity** component should be engineered directly into the model.
6. The methodology should be easily implemented by **IT**, so ideally **fast & easily calibratable** in (for example in closed form).
7. The model should **enhance models** that people are familiar with.

# Pricing Call Options Under Different Conventions

1. Classic Black & Scholes model (eg: for Equities)

$$\boxed{\begin{aligned} C(S_0,t) &= e^{-r(T-t)}[FN(d_1) - KN(d_2)] \\ d_1 &= \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_0}{K}\right) + (r - q + \tfrac{1}{2}\sigma^2)(T-t)\right] \\ d_2 &= d_1 - \sigma\sqrt{T-t} \end{aligned}} \quad (1)$$

2. Garman Kohlhagen model (eg: for FX)

$$\boxed{\begin{aligned} C(S_0,t) &= S_0 e^{-r_f(T-t)}N(d_1) - Ke^{-r_d(T-t)}N(d_2) \\ d_1 &= \frac{1}{\sigma\sqrt{(T-t)}}[\ln(S_0/K) + (r_d - r_f + \tfrac{1}{2}\sigma^2)(T-t)] \\ d_2 &= d_1 - \sigma\sqrt{(T-t)} \end{aligned}} \quad (2)$$

3. Options on Normal Underlyings (eg: for negative interrest rates)

$$\boxed{\begin{aligned} C(S_0,t) &= e^{-r(T-t)}[(F-K)N(d) + \sigma\sqrt{T-t}N'(d)] \\ d &= \frac{F-K}{\sigma\sqrt{T-t}} \end{aligned}} \quad (3)$$

# Price to Implied Vol: Bisection, Newton Raphson, Brent

**Algorithm 1** Bisection return $\sigma_i$ given $P, S_t, K, r_d, r_f, r, q, T$

**Ensure:** $P \approx Pricer(S_t, K, \sigma_i, T, r_d, r_f, q, r, T)$

$\epsilon \leftarrow 0.01$; $N = 50$; $\sigma_+ \leftarrow 3.0$; $\sigma_- \leftarrow 0.01$;
**for** $i = 1$ to $N$ **do**
    $\sigma_i \leftarrow \frac{\sigma_+ + \sigma_-}{2}$
    **if** $P > Pricer(S_t, K, \sigma, T, r_d, r_f, q, r)$ **then**
        $\sigma_+ \leftarrow \sigma_i$
    **else**
        $\sigma_- \leftarrow \sigma$
    **end if**
    **if** $\left|P - Pricer(S_t, K, \sigma, T, r_d, r_f, q, r)\right| < \epsilon$ **then**
        $i \leftarrow N$
    **end if**
**end for**
**Return** $\sigma_i$

# Implied Volatility & Market Prices: Classic Representation

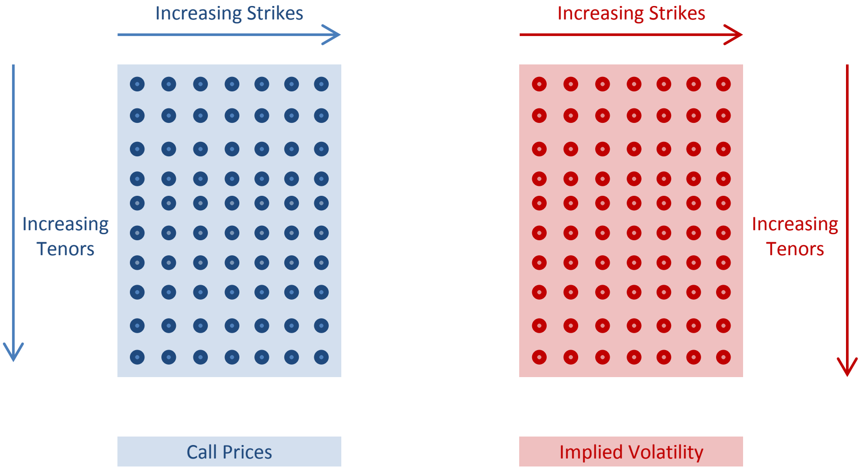



Figure: Visualization for the core simple de-arbing idea

# The Implied Volatility: a Continuous 3D Structure

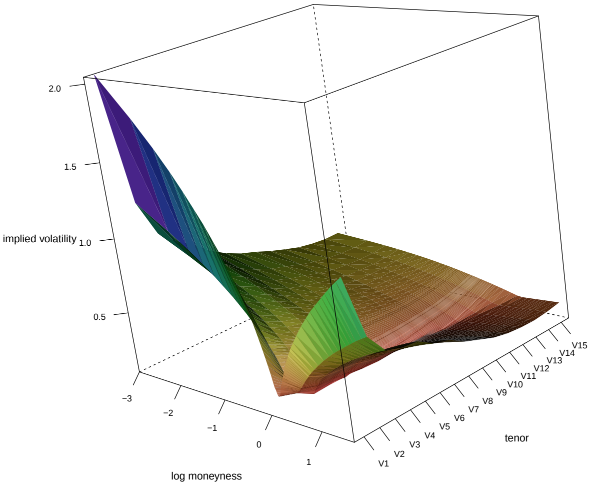



Figure: Arbitrageable Vol (closest arb-free mirror in figure 41)

# Implied Volatility Surface & Dimensionality

1. Why do we need to reduce **dimensionality**? (infinite granularity means infinite dimenssionality)

$$\Sigma = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \sigma_{80\%,30d} & \cdots & \sigma_{100\%,30d} & \cdots & \sigma_{120\%,30d} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \sigma_{80\%,60d} & \cdots & \cdots & \cdots & \sigma_{120\%,60d} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \sigma_{80\%,2Y} & \cdots & \cdots & \cdots & \sigma_{120\%,2Y} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

2. How do we handle so **many risk factors**? How to you **interpolate**? **extrapolate**?
3. How is **liquidity** engineered in the model itself?
4. How can we insure that the volatility is **arbitrage** free?

# Introducing the IVP & Recall on the SVI Model

1. **Parameterizing** the volatility surface with a Stochastic Vol Inspired (SVI) like model is very useful because, **interpolation and extrapolation** methodologies are **no longer required**[1], it allows for arbitrage detection and we can engineer liquidity costs directly in the model.
2. **Intuition** behind parameterizing the volatility surface: the function $f_1(k) = a$, would represent a volatility with a flat smile, $f_2(k) = a + b|k|$, would represent a vol of vol adjustment to $f_1$, $f_3(k) = a + b|k| + b\rho k$, would represent a skew adjustment to $f_2$ etc ...
3. We can **build abstraction** in this manner by adding parameters to fine tune the dynamics of the volatility surface. We can add as many parameters as one wishes bearing in mind that the aim is still for the model to remain **parsimonious**.

[1] on the strike axis (moneyness/logmoneyness, delta axis), but still required on the tenor axis

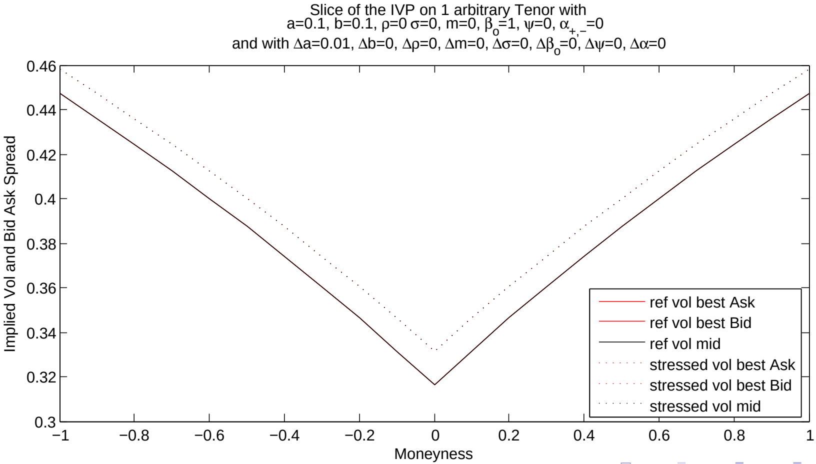
# SVI & IVP: the Parameters

The IVP model has for skeleton the celebrated raw SVI:

1. $a_\tau$ adjusts the **vertical** displacement of the smile
2. $b_\tau$ adjusts the **angle** between left and right asymptotes
3. $\rho_\tau$ adjusts the **orientation** of the graph
4. $m_\tau$ is the **horizontal** displacement of the smile
5. $\sigma_\tau$ adjusts the smoothness of the **vertex**
6. $\beta_\tau$ the **downside** transform for making the wings sub-linear
7. $\alpha_{0,\tau}$ & $\alpha_\tau(.)$ are the infinitesimal & function **ATM bid-ask** position size adjustment
8. $\psi_{0,\tau}$ & $\psi_\tau(.)$ are the infinitesimal & function **wing curvature bid-ask** position size adjustment
9. $\eta_{\alpha,\tau}$ & $\eta_{\psi,\tau}$ are the ATM & wing curvature bid-ask market impact **elasticity (or liquidity horizon)** respectively.
10. $p_\tau$ is the **position** size
11. $\lambda_{1,\tau}$ & $\lambda_{2,\tau}$ the in-between tenors "interpolators"

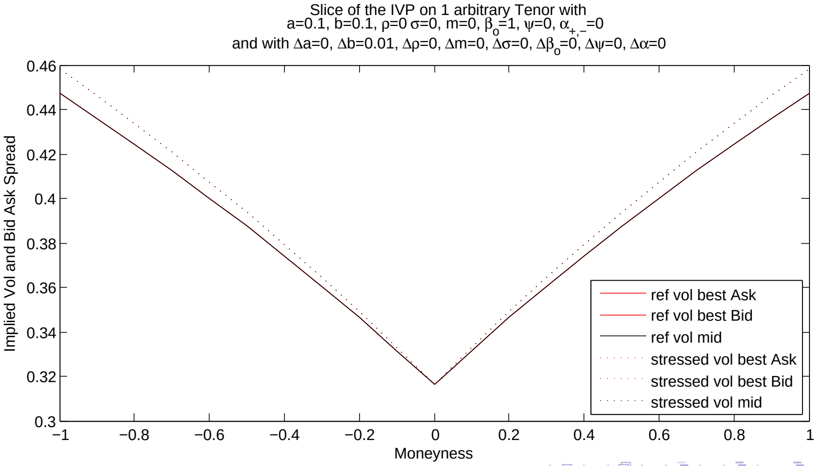
# Change of the $a$ Parameter in the Raw SVI

$$\sigma^2_{o,\tau}(k) = a_\tau + b_\tau\left[\rho_\tau(k-m_\tau) + \sqrt{(k-m_\tau)^2+\sigma^2_\tau}\right]$$



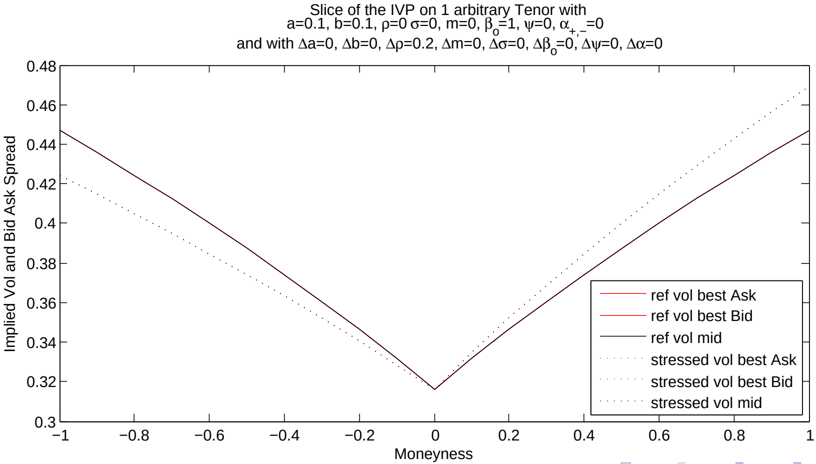
# Change of the $b$ Parameter in the Raw SVI

$$\sigma^2_{o,\tau}(k) = a_\tau + b_\tau \left[ \rho_\tau (k - m_\tau) + \sqrt{(k - m_\tau)^2 + \sigma^2_\tau} \right]$$



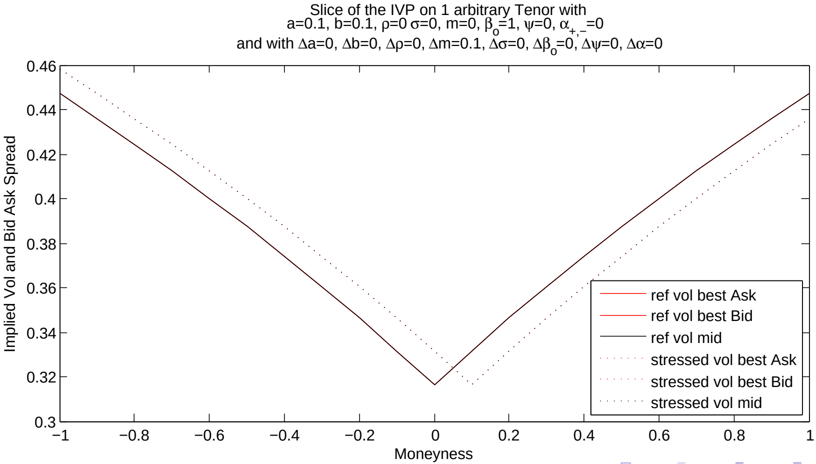
# Change of the $\rho$ Parameter in the Raw SVI

$$\sigma^2_{o,\tau}(k) = a_\tau + b_\tau \left[ \rho_\tau (k - m_\tau) + \sqrt{(k - m_\tau)^2 + \sigma_\tau^2} \right]$$



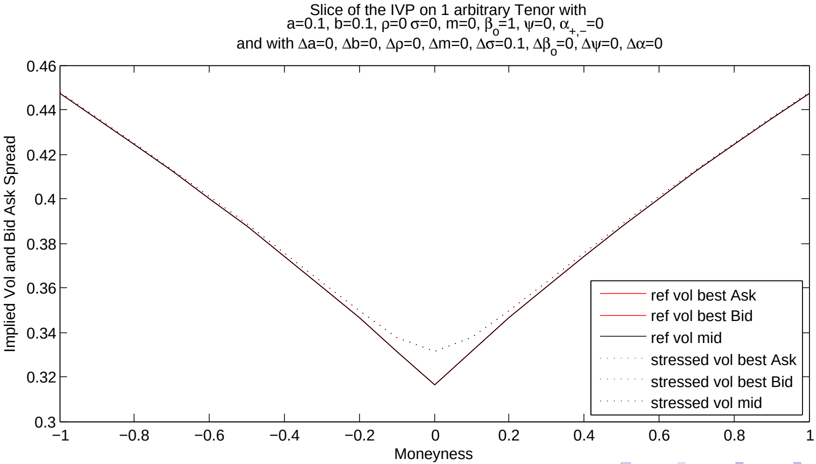
# Change of the $m$ Parameter in the Raw SVI

$$\sigma^2_{o,\tau}(k) = a_\tau + b_\tau\left[\rho_\tau\left(k - m_\tau\right) + \sqrt{\left(k - m_\tau\right)^2 + \sigma^2_\tau}\right]$$



# Change of the $\sigma$ Parameter in the Raw SVI

$$\sigma^2_{o,\tau}(k) = a_\tau + b_\tau\left[\rho_\tau (k - m_\tau) + \sqrt{(k - m_\tau)^2 + \sigma_\tau{}^2}\right]$$



# IVP: the Parameters

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11. $\lambda_{1,\tau}$ & $\lambda_{2,\tau}$ the in-between tenors "interpolators"

# IVP: the Full Formula

1. The main formula is presented below:

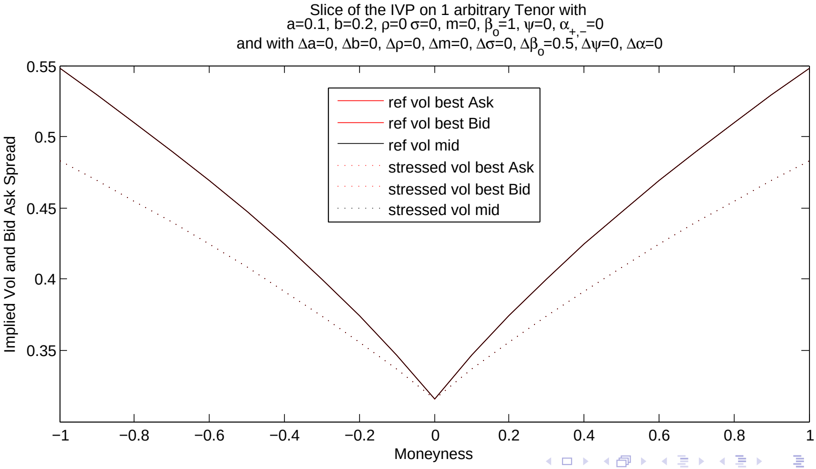
$$
\begin{aligned}
\sigma^2_{o,\tau}(k) &= a_\tau + b_\tau \left[ \rho_\tau (z_{o,\tau} - m_\tau) + \sqrt{(z_{o,\tau} - m_\tau)^2 + \sigma_\tau{}^2} \right] \\
z_{o,\tau} &= \frac{k}{\beta_{o,\tau}{}^{1+4|k-m|}} \\
\sigma^2_{+,\tau}(k,p) &= a_\tau + b_\tau \left[ \rho_\tau (z_{+,\tau} - m_\tau) + \sqrt{(z_{+,\tau} - m_\tau)^2 + \sigma_\tau{}^2} \right] + \alpha_\tau(p) \\
z_{+,\tau} &= z_{o,\tau}[1 + \psi_\tau(p)] \\
\sigma^2_{-,\tau}(k,p) &= a_\tau + b_\tau \left[ \rho_\tau (z_{-,\tau} - m_\tau) + \sqrt{(z_{-,\tau} - m_\tau)^2 + \sigma_\tau{}^2} \right] - \alpha_\tau(p) \\
z_{-,\tau} &= z_{o,\tau}[1 - \psi_\tau(p)] \\
\alpha_\tau(p) &= \alpha_{0,\tau} + (a_\tau - \alpha_{0,\tau})(1 - e^{-\eta_{\alpha_\tau} p}) \\
\psi_\tau(p) &= \psi_{0,\tau} + (1 - \psi_{0,\tau})(1 - e^{-\eta_{\psi_\tau} p})
\end{aligned}
$$

2. Additional parameters ($\lambda_1$, $\lambda_2$) around interpolation in tenor space is presented in the paper[2].

[2] http://onlinelibrary.wiley.com/doi/10.1002/wilm.10422/abstract

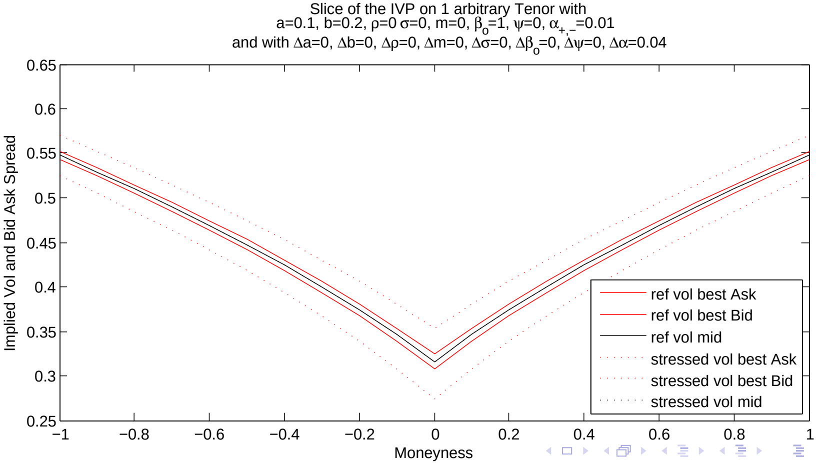
# Change of the $\beta$ Parameter in the IVP

$$z_{o,\tau} = \frac{k}{\color{red}{\beta_{o,\tau}}^{1+4|k-m|}}$$



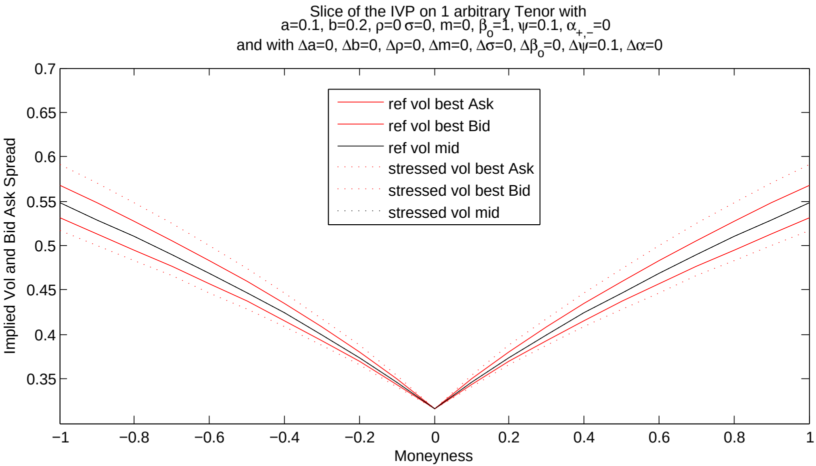
# Change of the $\alpha$ Parameter in the IVP

$$\sigma^2_{+,\tau}(k,p) = a_\tau + b_\tau\left[\rho_\tau\left(z_{+,\tau} - m_\tau\right) + \sqrt{\left(z_{+,\tau} - m_\tau\right)^2 + \sigma_\tau^2}\right] + \alpha_\tau(p)$$



# Change of the $\psi$ Parameter in the IVP

$$z_{+,\tau} = z_{o,\tau}\left[1 + \psi_\tau(p)\right]$$



# IVP: Calibration & Objective Function

1. We make sure that for every pillar tenor and every pillar strikes the relevant points are on average[3] as close as possible to the points induced by the parameterised version of the vol as defined by the IVP model.

$$\boxed{\begin{array}{l}\text{solve:}\\ \hat{\sigma}_t(\tau,d) = \underbrace{\arg\min}_{\tilde{\sigma}_t(\tau,d)} \sum_\tau \sum_d \left[\sigma_{i,t}(\tau,d) - \tilde{\sigma}_t(\tau,d)\right]^2\end{array}} \quad (4)$$

2. The calibration becomes an optimization by constraint problem (explained in section 2).
3. Calibration can be done both numerically and in closed form as we will see next.

[3] Euclidean distance

# IVP: Closed Form Calibration

For example if $\sigma = 0$, $\beta = 1$ and $p \approx 0$[4] we get:

$$
\begin{aligned}
\sigma^2_{+,\tau}(k) &= a_\tau + b_\tau\Big(\rho_\tau[k(1+\psi_\tau) - m] + |k(1+\psi_\tau) - m|\Big) + \alpha_\tau \\
\sigma^2_{o,\tau}(k) &= a_\tau + b_\tau\Big(\rho_\tau(k-m) + |k-m|\Big) \\
\sigma^2_{-,\tau}(k) &= a_\tau + b_\tau\Big(\rho_\tau[k(1-\psi_\tau) - m] + |k(1-\psi_\tau) - m|\Big) - \alpha_\tau \\
\hat{m}_\tau &= \underbrace{\arg\min}_{k} \sigma^2_{o,\tau}(k) \\
\hat{a}_\tau &= \sigma^2_{o,\tau}(m) \\
\hat{b}_\tau &= \frac{\sigma^2_{o,\tau}(m+k) + \sigma^2_{o,\tau}(m-k) - 2\hat{a}_\tau}{2|k|} \\
\hat{\rho}_\tau &= \frac{\sigma^2_{o,\tau}(m+k) - \sigma^2_{o,\tau}(m-k)}{2\hat{b}_\tau k} \\
\hat{\alpha}_\tau &= \sigma^2_{+,\tau}(m) - \hat{a}_\tau + m\hat{b}_\tau\hat{\rho}_\tau + |m|\hat{b}_\tau \\
\hat{\psi}_\tau &= \frac{\sigma^2_{+,\tau}(m) + \sigma^2_{-,\tau}(m) - 2\hat{a}_\tau}{2|m|\hat{b}_\tau}
\end{aligned}
$$

[4] these constraints are only there to make the results simple for the sake of the presentation only for illustration purpose (note that knowing $\eta_\alpha$ & $\eta_\psi$ become obsolete)

1. Whether it is in trading or in risk, being able to formalize the quantitative problem in terms of distribution allows us to have a better **certainty index** (in Statistical Arbitrage) or a better **measure of risk** (in Risk Management).
2. The conditional bumping formula is defined by equation (5) where we define the parameters in red as primary and in blue as secondary.

$$\boxed{dX_t = \frac{\theta_{\tau,t}}{\kappa_\theta}(\mu_{\tau,t} - X_t)dt + \kappa_W \sigma X_t^\alpha (1 - X_t^2)^\beta dW_t} \tag{5}$$

3. The conditional bumping equation (5) models all risk factors in the context of this presentation: Spot, Forward, Interest Rate, Implied Vol (eg: skew traders), etc ...

# Stat Arb & Risk Management: the Parameters

**Primary parameters**

1. $\theta_t$, the rolling speed of mean reversion
2. $\mu_t$, the long term rolling mean
3. $\alpha$, the positivity flag enforcer
4. $\beta$, the $[-1, +1]$ boundary flag enforcer
5. $\{\bigcup dW_i\}_{i=t-\tau}^{t}$, the set of historical deviations given the assumed model.

**Secondary parameters**

1. $\tau$, the rolling window length of calibration.
2. $\kappa_\theta$, the speed of mean reversion dampener,
3. $\kappa_W$, the variance enhancer.

# Stat Arb & Risk Management: the Diffusions

1. Proportional bump (log-normal diffusion). Simply enforce $\theta = 0$, $\alpha = 1$, $\beta = 0$,
2. Absolute bumps (normal diffusion). Simply enforce $\theta = 0$, $\alpha = 0$, $\beta = 0$,
3. Mean reverting bumps where we enforce positivity (like in the case of the CIR diffusion),
4. Mean reverting bumps where we do not enforce positivity (like in the case of the OU diffusion). This one is particularly handy in defining the dynamics of the horizontal displacement $m$.
5. Mean reverting bumps bounded in $[-1, 1]$. For example the dynamics of the $\rho$ parameter in the SVI/gSVI/IVP. This one is particularly handy in defining the dynamics of the **skew** parameter $\rho$.

# Stat Arb & Risk Management: Diffusions Benefits

1. It is **versatile**: it models all the known risk models on top of new ones,
2. It is **deployable** and **robust**: once the calibration has been performed the same code works for every risk factor,
3. It is **leading**: it allows for anticipation in the regime change as opposed to waiting passively for responding to a regime change,
4. It is more **realistic**: when Vols (or interest rates) are high applying relative shifts overestimates the moves on the upside but underestimate the moves on the downside (same for some parameters),
5. It decreases **arbitrages** scenarios: since the diffusion of equation (5) is more realistic with respect to market observable phenomenon, the number of arbitrage opportunities in the stressed scenario generations decreases drastically when the conditional methodology is used as opposed to relative shifts especially when it comes to skew like strategies (eg: butterfly, call spreads etc ...).

# Stat Arb & Risk Management: Parameters Diffusions (eg)

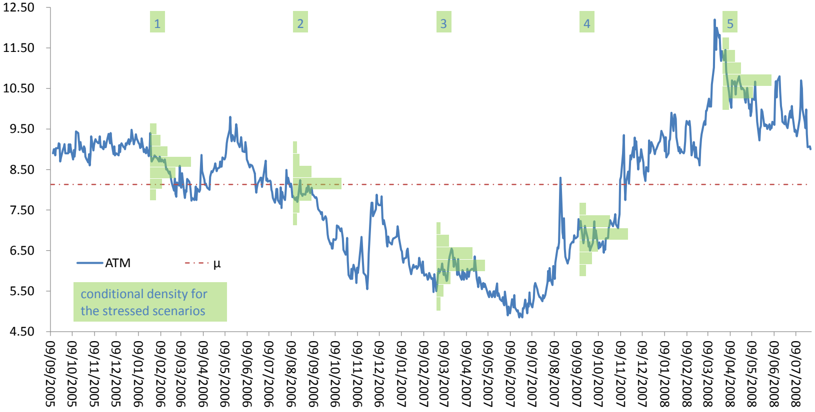
1. We want to consider the **co-movement between the risk factors** and spot as well as with themselves.
2. We may want to consider a conditional probability bumping on the parameters themselves and use more quant weaponry to address the **asymmetric** property of options:

$$\begin{aligned}
da_t &= \theta_a(\mu_a - a_t)dt + \sigma_a a_t dH_t^a \\
db_t &= \theta_b(\mu_b - b_t)dt + \sigma_b b_t dH_t^b \\
d\rho_t &= \theta_\rho(\mu_\rho - \rho_t)dt + \sigma_\rho(1 - \rho_t^2)dH_t^\rho \\
dm_t &= \theta_m(\mu_m - m_t)dt + \sigma_m m_t dH_t^m \\
&\ldots
\end{aligned}$$

3. Do we need to get to the closest **arbitrage free vol** for the cleansed scenarios?

# Stat Arb & Risk Management: Visual Example

$$d\sigma_t^{6M,ATM} = \theta_t^{6M,ATM}(\mu - \sigma_t^{6M,ATM})dt + \sigma_t^{6M,ATM} dH_t^{6M,ATM}$$



# Options Arbitrages: Calendar & Butterfly Rules

1. There are 2 real conditions that are important in the business context: there should not be any arbitrage opportunities in the strike and tenor domains.
2. A call with a higher strike should never be more expensive than a call of lower strike at the same maturity level. Equation (6), which represents the **Butterfly**[5] condition formalizes this idea.

$$\boxed{\forall \Delta, C(K-\Delta) - 2C(K) + C(K+\Delta) > 0} \tag{6}$$

3. A call with a shorter expiry should never be more expensive than a call of the same strike but of longer expiry. Equation (7), which represents the **Calendar** condition formalizes this idea.

$$\boxed{C(K, T+\Delta) - C(Ke^{-r\Delta}, T) > 0} \tag{7}$$

[5] we can alternatively use the call spread inequality

# Arb check on the Strike Axis: a Common Error

1. Note that interpolation simple does not insure the volatility surface to be arbitrage free even with the most advanced methodologies[6].
2. The Roger Lee condition is necessary **but not sufficient** and hence the results given by $b(1+|\rho|) \leq \frac{4}{T}$ from the SVI was elegant but ultimately incomplete.

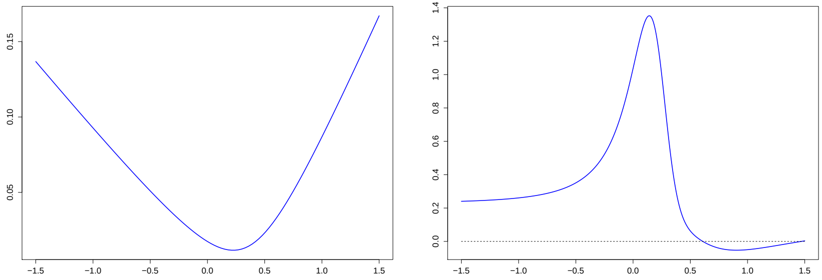



Figure: Example in which a linear interpolation yields an arbitrage

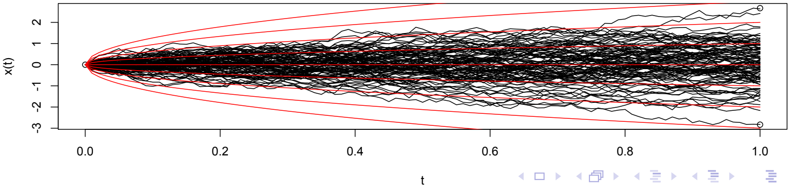
[6]Roger Lee condition $|\partial_k \sigma(k)| < 4/T$ is necessary but unfortunately **not sufficient**

# Arb Check on the Tenor Axis: Falling Variance

1. The Falling variance formula, of equation (8), will insure that extrapolation is done consistently with respect to the calendar spread arbitrage condition described by equation (7).

$$\boxed{\forall\, T > t, \forall\, k, \sigma^2_{T,k} \geq \sigma^2_{t,k}\frac{t}{T}} \tag{8}$$

2. Generally speaking extrapolation along the tenor axis in variance space is considered for the falling variance condition widely accepted.
3. We can see below a plot of 100 simulations of brownian motions generated and their $\sigma$'s (*the red lines widen as the expiry increases*).



# Additional Arbitrages in FX: triangle representation

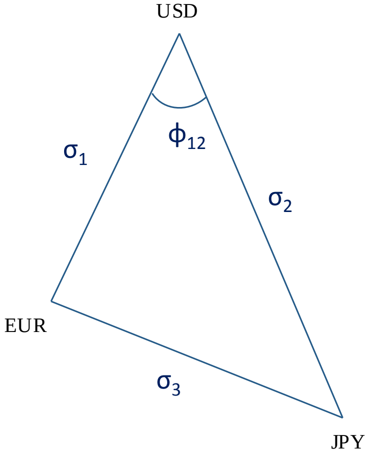



Figure: FX triangle representation for Spot/Forward and Implied Vol

# Arbitrages in the FX Market: Triangle Rules

1. If we associate the subscripts of $\frac{\text{EUR}}{\text{USD}}$, $\frac{\text{USD}}{\text{JPY}}$, and $\frac{\text{JPY}}{\text{EUR}}$ to the ones of figure 5 we have:
2. Triangle rule on spot/forward defined by equation (9) will be **disregarded** in the context of this presentation because non-USD crosses will be inferred from USD currency pairs.

$$
\begin{aligned}
S_1 &= S_2 \times S_3 \\
F_1 &= F_2 \times F_3
\end{aligned}
\tag{9}
$$

3. The triangle rule with respect to implied volatility defined by equation (11) will be **disregarded** because of benefits to complexity ratio reasons.

$$
\sigma_1 + \sigma_2 + \sigma_3 > 2 \max(\sigma_1 + \sigma_2 + \sigma_3) \tag{10}
$$

# Arbitrages in the FX market: Implied Correlation Rule

1. The triangle rule with respect to implied volatility defined by equation (11) will be **disregarded** because of benefits to complexity ratio reasons.

$$\sigma_1 + \sigma_2 + \sigma_3 > 2\max(\sigma_1 + \sigma_2 + \sigma_3) \tag{11}$$

2. Implied correlation is defined by equation (12) with the subscripts of figure 5.

$$\rho_{1,2} = \frac{\sigma_3^2 - \sigma_2^2 - \sigma_1^2}{2\sigma_1\sigma_2} = \cos\phi_{1,2} \tag{12}$$

3. One other condition is for the implied correlation matrix of equation (13) to be positive semi-definite[7].

$$\mathbb{C} = \begin{pmatrix} 1 & \rho_{1,2} & \cdots & \rho_{1,n} \\ \rho_{2,1} & 1 & \cdots & \rho_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n,1} & \cdots & \rho_{n,n-1} & 1 \end{pmatrix} \tag{13}$$

[7] usually disregarded

# Arbitrages in the FX Market: Implied Correlation

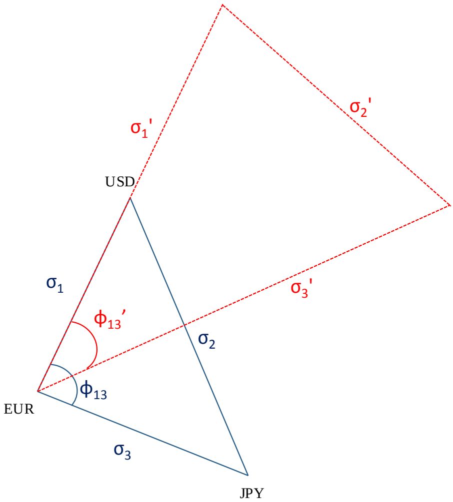



Figure: Representation exposing relationship between implied correlation and vol

# Dearbing: About the Importance of De-Arbing

1. Each vol surface, whether quoted or simulated, conveys information about the price and risk of a portfolio and it must therefore be realistic.
2. We wish to avoid the possibility that arbitrageable scenarios are driving our IM calculation
3. Dearbing is cumbersome so we can improve the scenario generation process to reduce the probability of arbitrages (we refer here to the presentation Mean Reverting Bumping: Application to Implied Volatility Historical Scenario Generation).
4. De-arbing is a convoluted mathematical optimization which perferct solution falls outside the scope of what we usually define to be a pragmatic benefits to complexity ratio so we propose, in this presentation, a partial de-arbing process.

# Dearbing: Optimization by Constraints

$$\boxed{\begin{array}{l}
\text{solve:} \\
\hat{\sigma}_t(\tau, d) = \underbrace{\arg\min}_{\tilde{\sigma}_t(\tau,d)} \sum_\tau \sum_d [\sigma_{i,t}(\tau, d) - \tilde{\sigma}_t(\tau, d)]^2 \\
\text{subject to:} \\
\forall d \in \{10, 25, 50, 75, 90\}, \\
0 < C\Big(K - \Delta, \sigma_0(K - \Delta, \tau)\Big) - C\Big(K, \sigma_0(K, \tau)\Big), \\
\forall \tau \in \{ON, 1W, 2W, 1M, 2M, 3M, 6M, 1Y, 18M, 2Y\}, \\
0 < C\Big(K, \tau + \Delta, \sigma_0(K, \tau + \Delta)\Big) - C\Big(Ke^{-r\Delta}, \sigma_0(Ke^{-r\Delta}, \tau)\Big)
\end{array}} \tag{14}$$

1. We make sure that for every pillar tenor and every pillar strikes the relevant points are mutually arbitrage free.
2. We try to find the shortest distances between the input vol and its closest arbitrage free mirror subject to the Call spread (equivalent to butterfly) and Calendar spread Conditions.

# De-Arbing: Objective Function

1. In order to use the usual optimization tools, we need to adjust the objective function to take in the constraints of the problem.
2. If we call $\mathcal{B}$ the call spread[8] arbitrage flag and $CS_1$ its impact in price.

$$CS_1 = \Big|C\Big(K-\Delta, \sigma_0(K-\Delta, \tau)\Big) - C\Big(K, \sigma_0(K, \tau)\Big)\Big| 1_{\mathcal{B}} \quad (15)$$

3. Let $\mathcal{C}$ be the Calendar spread arbitrage flag and $CS_2$ its price impact.

$$CS_2 = \Big|C\Big(K, \tau+\Delta, \sigma_0(K, \tau+\Delta)\Big) - C\Big(Ke^{-r\Delta}, \sigma_0(Ke^{-r\Delta}, \tau)\Big)\Big| 1_{\mathcal{C}} \quad (16)$$

4. Let's call $\mathcal{K}$ the constraint scalar[9], the objective function is adjusted as described in equation (17).

$$\hat{\sigma}_t(\tau, d) = \underbrace{\arg\min}_{\tilde{\sigma}_t(\tau,d)} \sum_{\tau} \sum_{d} [\sigma_{i,t}(\tau, d) - \tilde{\sigma}_t(\tau, d)]^2 + \mathcal{K}(CS_1 + CS_2) \quad (17)$$

[8] equivalent to the Butterfly condition

[9] a big enough number to make sure the constraints are respected but not too big to create numerical instabilities

# De-Arbing: an Imperfect Simplified Implementation

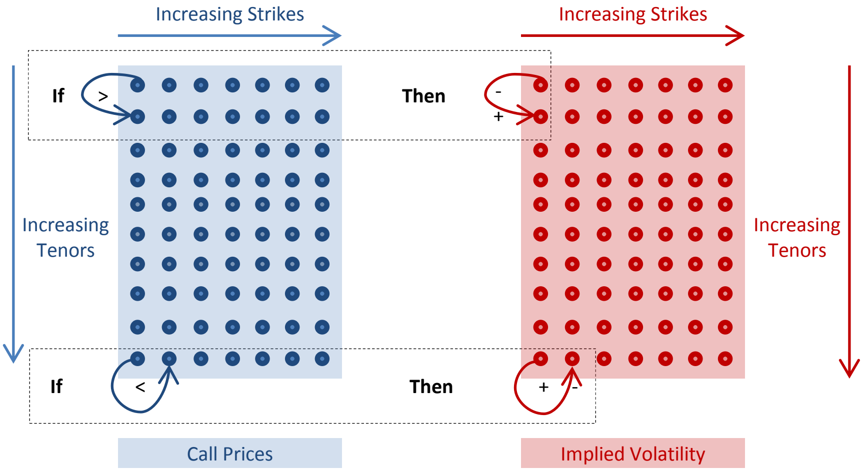



Figure: Visualization for the core simple de-arbing idea

# De-arbing: be Careful with FX Data

1. The data in FX is listed in delta space but the algorithm from slide 51 assumes that the data is conveniently aligned in log-moneyness space.

$$\Delta_f = \phi e^{-r f t} N(\phi \frac{1}{2} \sigma \sqrt{t}) \tag{18}$$

2. The market delta space pillars are the 10, 25, 50, 75, 90[10] delta.
3. Generally speaking interpolation will be done **linearly in variance** space as opposed to volatility space and extrapolation will be flat in volatility space. There is also the possibility to perform a cubic spline.
4. The pillars for our tenor axis have been qualitatively chosen to be: ON, 1W, 2W, 1M, 2M, 3M, 6M, 1Y, 18M, 2Y. The delta to log-moneyness conversion creates increasing mis-alignments as the tenor increases.

[10] discussion around have information on the 1 and 99 delta are currently happening

# De-arbing: be Careful with FX Data

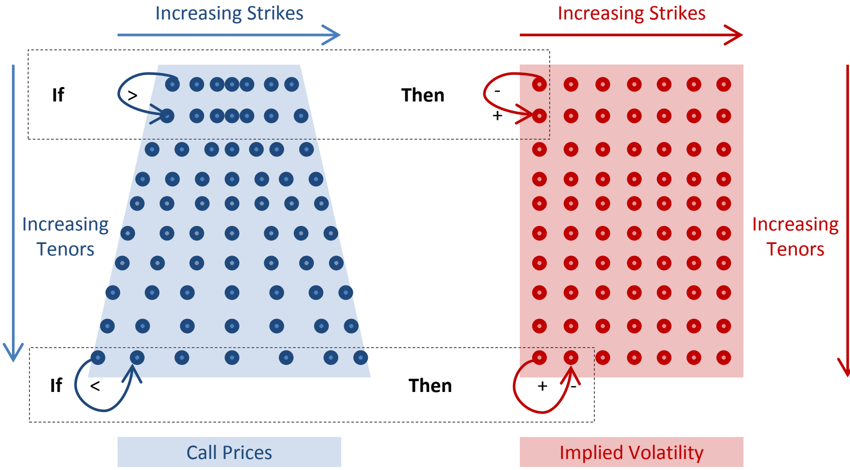



Figure: Visualization for the core simple de-arbing idea approximation

# Example of De-Arbed Closest Vol

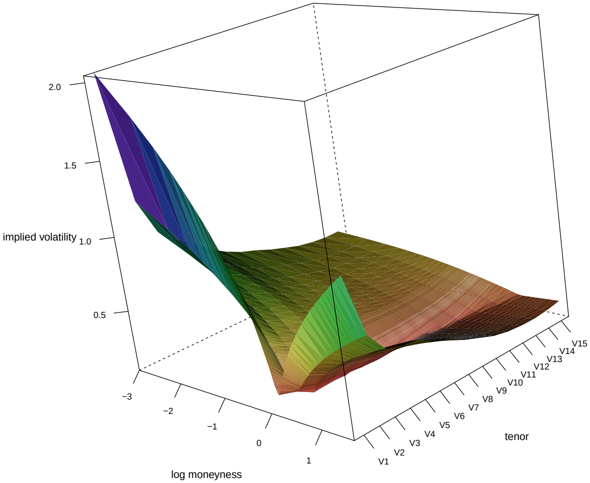



Figure: Closest arbitrage free vol of figure 6.