Convergence of Heston to SVI Proposed Extensions:
Rational & Conjecture for the Convergence of Extended Heston to the Implied Volatility surface Parametrization
(working paper)

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September 18, 2017

Abstract

A mathematical and a market argument on the sub-linearity of the wings for the implied variance is given. Gatheral stochastic volatility inspired (SVI) parameterization claim to have two key properties that have led to its subsequent popularity with practitioners is exposed. Namely the linearity in the log-strike k as $|k| \to \infty$ consistent with Roger Lees moment formula [9] as well as its connection to the Heston model [5] are examined more in details. Though correct, the former point led to the model subsequent decommission in the industry. We explain this apparent contradiction by pointing to a mathematically convenient [1] chosen factor in the Heston model which we expose and consequently introduce couple candidates: the p-Heston and the Inferred Correlation [10] models instead. The link between the latter and the SVI being broken, we propose a connection to the Implied Volatility surface Parametrisation (IVP) [11, 12] recently introduced and propose a conjecture between a mirror convergence towards these models using the parallel between the traditional Heston to SVI convergence [5].

Keywords: Volatility Smile, Implied Volatility Wings, Heston model, p-Heston, IVP, SVI, Stochastic Volatility, Implied Volatility surface Parametrization, Asymptotic Convergence, Local Correlation Surface.

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1 Introduction

1.1 Context

Gatheral introduced the stochastic volatility inspired (SVI) parameterization at Merrill Lynch in 1999 [5]. The parameterization was claimed to have two key properties that have led to its subsequent popularity with practitioners, the ones relevant to this paper being that:

- for a fixed time to expiry T, the implied Black-Scholes variance $\sigma^2_{BS}(k,T)$ is linear in the log-strike k as $|k| \to \infty$ consistent with Roger Lees moment formula [9]
- the model converges asymptotically to the Heston model [5].

Concurrently we have seen the academic rise and industry demand for data driven models as opposed to models that are just mathematically convenient.

1.2 Problem Formulation

On the options markets, we seldom observe wings that are sub-linear (eg: figure 7). This latter point led to the SVI being decommissioned at Merrill Lynch (then Bank of America Merrill Lynch) because of its poor fit in the wings. The Heston model itself is infamous for being poor at modeling the accurately of the smile. This means that two of the most famous financial models are in fact too inaccurate to be still useful in the context of precise pricing. We can therefore legitimately ask ourselves these two questions:

- What are the mathematical arguments that induces the observed sub-linearity of the wings?
- How can we reconcile or potentially enhance the SVI and/or the Heston model without totally changing the underlying models so as to make them corroborate more with observed data?

1.3 Agenda

We will roughly try to organize this paper in 5 sections:

- In section 2 we summaries the SVI and its extensions. More specifically, in subsection 2.4 we will recall and propose the newly introduced Implied Volatility surface Parametrisation (IVP) as a natural extension of the raw SVI model engineered to address the limitations of the latter in the wings. In this context we take the opportunity, in subsection 2.3 to recall and explore the transform functions between the natural and the raw SVI using the results from Gatheral and Jacquier [7]. The latter section is motivated by the necessity of showing equivalence since the IVP is an extension of the raw SVI but the literature available dealing with the convergence between the Heston and the SVI is done via its natural version [5] and not its raw version.

- In section 3 we explore the confusing road towards understanding the relationship between finite moments and wings' sub-linearity with the classic mathematical assumptions and models. More specifically we will delve into the spot focus argument in subsection 3.1 and propose in subsection 3.2 the p-Heston vol focused argument as a natural extension.
- In section 4 we will recall the Inferred Correlation model in order to introduce the local correlation concept to provide an alternative model, this time not spot but correlation focused.
- Finally we include 4 appendices to help complete the paper namelly:
  - In appendix A we will recall the convergence results between the natural SVI to the Heston model using Gatheral's and Jacquier's work [5].
  - In appendix B, we include the main points of the proof for the right and left wing sub-linearity.
  - In appendix C, we present more formally the other parameters of the IVP.

2 The SVI and its IVP extension

We refer here to the work done by Jaquier and Gatheral on Arbitrage Free SVI in [7] in which the correspondence between the raw and natural SVI model is derived. We will however summarize the main points next.

Remark In terms of notations and in the foregoing, we consider a stock price process $(S_t)_{t\geq 0}$ with natural filtration $(\mathcal{F}_t)_{t\geq 0}$, and we define the forward price process $(F_t)_{t\geq 0}$ by $F_t := \mathbb{E}(S_t|\mathcal{F}_0)$. For any $k \in \mathbb{R}$ and $t > 0$, $C_{\mathrm{BS}}(k, \sigma^2 t)$ denotes the Black-Scholes price of a European Call option on S with strike $F_t e^k$, maturity t and volatility $\sigma \geq 0$. We shall denote the Black-Scholes implied volatility by $\sigma_{BS}(k,t)$, and define the total implied variance by

$$w(k, \chi_R) = \sigma_{\mathrm{BS}}^2(k,t)t.$$

The implied variance v shall be equivalently defined as $v(k,t) = \sigma_{\mathrm{BS}}^2(k,t) = w(k,t)/t$. We shall refer to the two-dimensional map $(k,t) \mapsto w(k,t)$ as the volatility surface, and for any fixed maturity $t > 0$, the function $k \mapsto w(k,t)$ will represent a slice.

2.1 The Raw Stochastic Volatility Inspired (SVI) model

2.1.1 History

One advertised[1] advantage of the SVI is that it can be derived from Heston [8, 6], a model used by many financial institutions for both Risk, Pricing and sometimes Statistical Arbitrage purposes. One of the main advantages of this parametrization is its simplicity. Advertised as being parsimonious, its parametrisation assumed linearity in the wings (which yields a poor fit in the wings) and no Bid Ask liquidity model which led to it being decommissioned couple of years after its birth mainly because of its inability to handle variance swaps.

2.1.2 Formula

For a given maturity slice, we shall use the notation $w(k, \chi_R)$ where $\chi_R = \{a, b, \rho, m, \sigma\}$ represents a set of parameters, and the t-dependence is dropped. For a given parameter set. Then the raw SVI parameterization of implied variance reads:

$$w(k, \chi_R) = a + b[\rho(k - m) + \sqrt{(k - m)^2 + \sigma^2}] \tag{1}$$

with k being the log-moneyness ($\log(\frac{K}{F})$ with F being the value of the forward).The advantage of Gatheral's model was that it was a parametric model that was easy to use, yet had enough complexity to properly model the volatility surface and its dynamic. Figure 3 illustrates the change in the ρ parameter (the skew risk), Figure 2 illustrates the change in the b parameter (the vol of vol risk), Figure 1 illustrates the change in the a parameter (the general volatility level risk), Figure 5 illustrates the change in the σ parameter (the ATM volatility risk) and finally Figure 4 illustrates the change in the m parameter (the horizontal displacement risk).

2.2 The natural SVI

There exist other forms of the SVI developed by Gatheral [7] which are closely linked to the raw SVI of section 2.1. The one relevant to the Heston to SVI convergence is its natural parameterization given by equation (2).

$$w(k, \chi_N) = \Delta + \frac{w}{2}\left\{1 + \zeta\rho(k - \mu) + \sqrt{(\zeta(k - \mu) + \rho)^2 + (1 - \rho^2)}\right\}, \tag{2}$$

where $w \geq 0, \Delta \in \mathbb{R}, |\rho| < 1$ and $\zeta > 0$.

[1] One of the main point of this paper is to expose a small mistake that was done in one particular paper [5] but for the sake of the introduction, we will make this remark as a footnote.

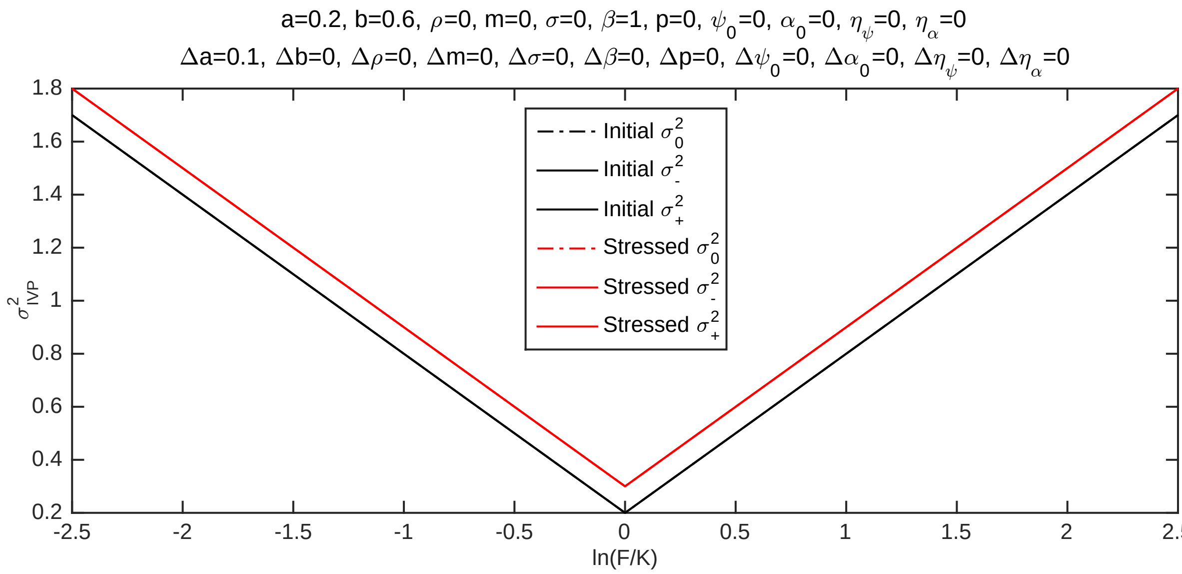


Figure 1: Impact of a change in the a parameter in the rawSVI/gSVI/IVP model

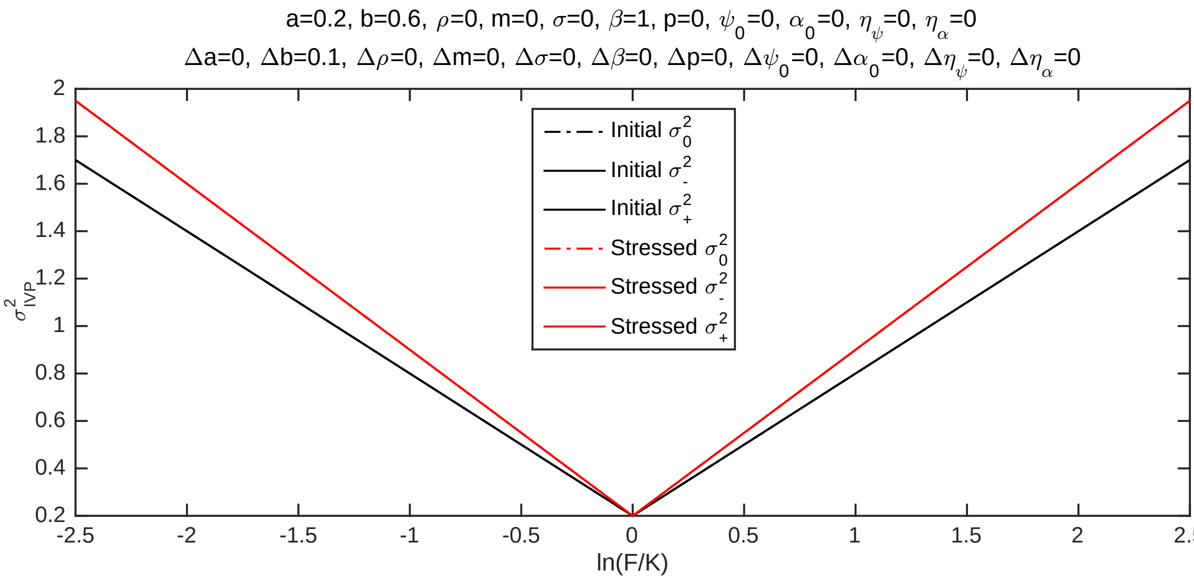


Figure 2: Impact of a change in the b parameter in the rawSVI/gSVI/IVP model

2.3 Transforms Raw and Natural SVIs

The IVP model that we introduce in section 2.4 is an extension of the raw SVI but the convergence between Heston and SVI [5] done in its natural version. We therefore introduce in this section couple of relevant transforms.

Remark We take this opportunity to introduce lemma *Raw to Natural SVI Transform* of equation 3 and lemma *Natural to Raw SVI Transform* of equation (4) that we use in *Theorem (SVI equivalence)*.

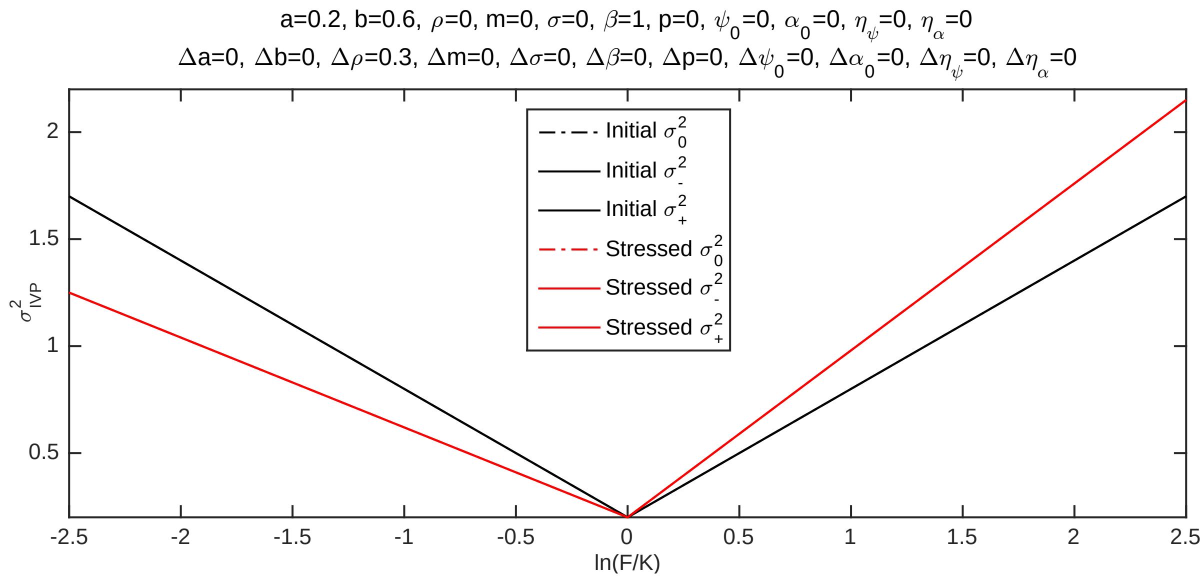


Figure 3: Impact of a change in ρ parameter in the rawSVI/gSVI/IVP model

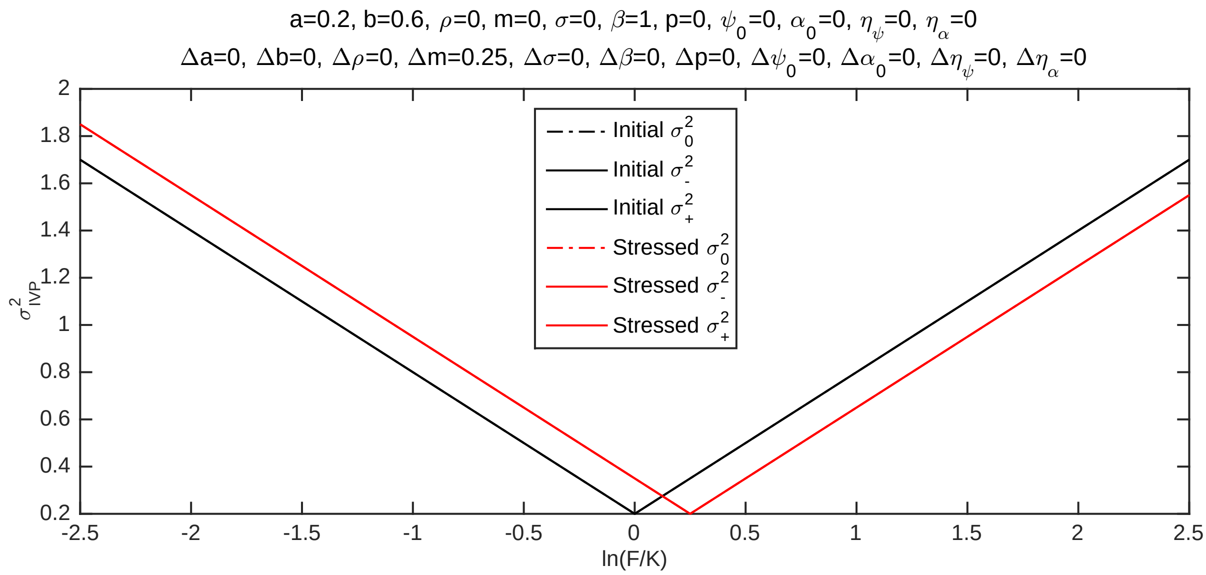


Figure 4: Impact of a change in m parameter in the rawSVI/gSVI/IVP model

Lemma (Raw to Natural SVI Transform): If we call $\chi_R = \{a, b, \rho, m, \sigma\}$ a given parameter set for the function $f \colon \mathbb{R}^{+,*} \times \mathbb{R}^{+} \times [-1, +1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^{+,*}$ given by $f(k, \chi_R) = a + b[\rho(k - m) + \sqrt{(k - m)^2 + \sigma^2}]$, and call $\chi_N = \{\Delta, \mu, \rho, w, \zeta\}$ a given parameter set for the function $g \colon \mathbb{R}^{+,*} \times \mathbb{R}^{+} \times [-1, +1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^{+,*}$ given by $g(k, \chi_N) = \Delta + \frac{w}{2}\left\{1 + \zeta\rho(k - \mu) + \sqrt{(\zeta(k - \mu) + \rho)^2 + (1 - \rho^2)}\right\}$, then we can define a set of 5 functions $\cup_{i=1}^{5} T_i$ mapping χ_R onto χ_N the following way:

$$(a, b, \rho, m, \sigma) = \left(\Delta + \frac{w}{2}(1 - \rho^2), \frac{w\zeta}{2}, \rho, \mu - \frac{\rho}{\zeta}, \frac{\sqrt{1 - \rho^2}}{\zeta}\right) \tag{3}$$

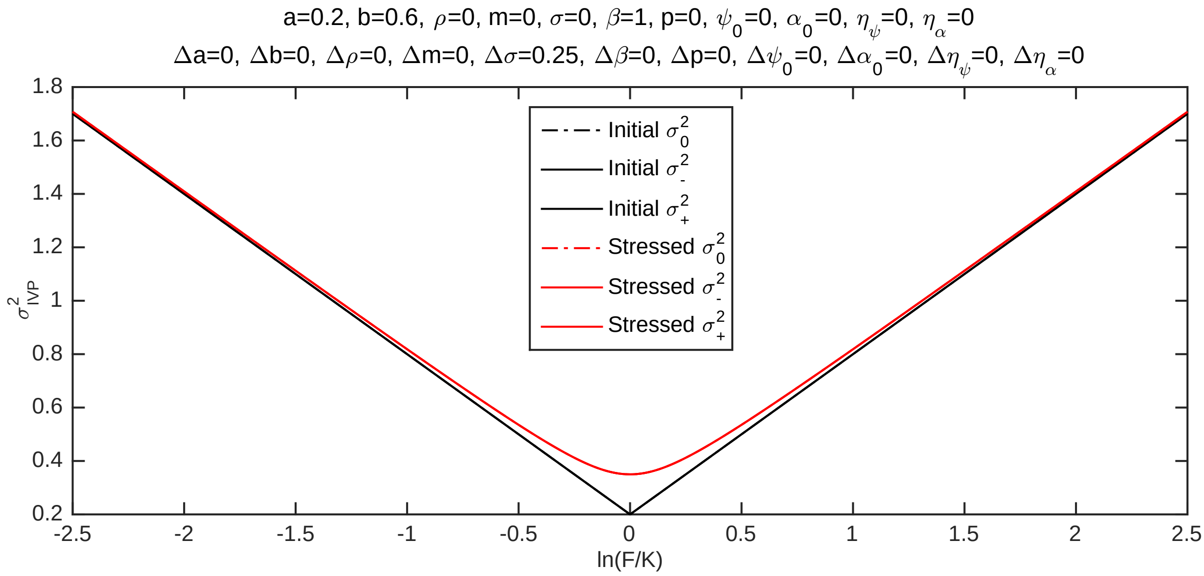


Figure 5: Impact of a change in σ parameter in the rawSVI/gSVI/IVP model

Proof Using equation (2) and rearranging it in the form of equation (1), we note that the ρ of χ_N is the same as χ_R, we can then solve the 4 other parameters sequentially noticing that $a = \Delta + \frac{w}{2}(1-\rho^2)$, $b = \frac{w\zeta}{2}$, $m = \mu - \frac{\rho}{\zeta}$ and $\sigma = \frac{\sqrt{1-\rho^2}}{\zeta}$.

Lemma (Natural to Raw SVI Transform): If we call $\chi_R = \{a, b, \rho, m, \sigma\}$ a given parameter set for the function $f\colon \mathbb{R}^{+,*} \times \mathbb{R}^{+} \times [-1,+1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^{+,*}$ given by $f(k, \chi_R) = a + b[\rho(k-m) + \sqrt{(k-m)^2 + \sigma^2}]$, and call $\chi_N = \{\Delta, \mu, \rho, w, \zeta\}$ a given parameter set for the function $g\colon \mathbb{R}^{+,*} \times \mathbb{R}^{+} \times [-1,+1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^{+,*}$ given by $g(k, \chi_N) = \Delta + \frac{w}{2}\left\{1 + \zeta\rho(k-\mu) + \sqrt{(\zeta(k-\mu)+\rho)^2 + (1-\rho^2)}\right\}$, then we can define a set of 5 functions $\cup_{i=1}^{5} T_i$ mapping χ_N onto χ_R the following way:

$$(\Delta, \mu, \rho, w, \zeta) = \left(a - b\sigma(1-\rho)^{3/2}, m + \frac{\rho\sigma}{\sqrt{1-\rho^2}}, \rho, \frac{2b\sigma}{\sqrt{1-\rho^2}}, \frac{\sqrt{1-\rho^2}}{\sigma}\right). \tag{4}$$

Proof Using equation (1) and rearranging it in the form of equation (2), we note that the ρ of χ_R is the same as χ_N, we can then solve the 4 other parameters sequentially noticing that $\Delta = a - b\sigma(1-\rho)^{3/2}$, $\mu = m + \frac{\rho\sigma}{\sqrt{1-\rho^2}}$, $w = \frac{2b\sigma}{\sqrt{1-\rho^2}}$ and $\zeta = \frac{\sqrt{1-\rho^2}}{\sigma}$.

Theorem (SVI equivalence) If we call $\chi_R = \{a, b, \rho, m, \sigma\}$ and $\chi_N = \{\Delta, \mu, \rho, w, \zeta\}$ a given set of parameters defining respectively the Raw SVI given by $\sigma_R^2\colon \mathbb{R}^{+,*} \times \mathbb{R}^{+} \times [-1,+1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^{+,*}$ given by $\sigma_R^2(k, \chi_R) = a + b[\rho(k-m) + \sqrt{(k-m)^2 + \sigma^2}]$, and $\sigma_R^2(k, \chi_N) = \Delta + \frac{w}{2}\left\{1 + \zeta\rho(k-\mu) + \sqrt{(\zeta(k-\mu)+\rho)^2 + (1-\rho^2)}\right\}$ respectively then $\sigma_R^2(k, \chi_R)$ and $\sigma_R^2(k, \chi_N)$ are equivalent functions.

Proof Using proof of equivalence, we can use lemma *Raw to Natural SVI Transform* to prove that $\sigma_R^2(k, \chi_N) \Rightarrow \sigma_N^2(k, \chi_R)$ and $\sigma_R^2(k, \chi_N) \Leftarrow \sigma_N^2(k, \chi_R)$ using lemma *Natural to Raw SVI Transform*, and therefore $\sigma_R^2(k, \chi_N) \Leftrightarrow \sigma_N^2(k, \chi_R)$.

2.4 Relation between IVP and raw SVI

Jim Gatheral developed the SVI model at Merrill Lynch in 1999 and implemented in 2005. The SVI was subsequently decommissioned in 2010 because of its limitations in accurately pricing out of the money variance swaps (for example short maturity Var Swaps on the Eurostoxx are overpriced when using the SVI). This is because the wings of the SVI are linear and have a tendency to overestimate the out of the money (OTM) variance swaps. Benaim, Friz and Lee [2] gave a mathematical justification for this market observation. The paper suggests that the implied volatility cannot grow asymptotically faster than $\sqrt{k}$ but may grow slower than $\sqrt{k}$ when the distribution of the underlier does not have finite moments (eg: has heavy tails). This suggest that the linear wings of the SVI model may overvalue really deeply OTM options which is observable in the markets.

2.4.1 Downside transform parameter

In order to address the limitations of the SVI model in the wings, while keeping its core skeleton intact, Mahdavi-Damghani [12] proposed a change of variable which purpose was to penalize the wings's linearity. The additional relevant parameter was called β and was later extended in order to also address the liquidity constraints of the model [11] especially given the challenging regulatory environment[2].

Remark Mahdavi-Damghani initially named the model gSVI[12] but renamed it IVP [11] when the liquidity parameters were incorporated. In the context of this paper, in which liquidity is not of central importance, we focus on the mid application of the downside transform, namely on the penalization of the wings so the gSVI and the IVP may be mentioned interchangeably.

In order to keep the number of factor limited, this β penalization functions was made symmetrical on each wing[3]. The function needed to be increasing as it gets further away from m and majored by a linear function increasing in $[m; +\infty[$ and decreasing in $]-\infty; m]$ and increasing in concavity the further away it gets from the center. Equation (5) summarizes the gSVI[4]. The penalization was given by equation (5b). Figure 6 illustrates

[2]e.g. FRTB

[3]But induced geometrically more significant on the steepest wing: for e.g. more significant on the left wing in the Equities market and more significant on the right wing of the Commodities (excluding oil) market

[4] or alternatively IVP's mid, model

the change in the β parameter.

$$\sigma^2_{gSVI}(k) = a + b\left[\rho(z-m) + \sqrt{(z-m)^2 + \sigma^2}\right] \tag{5a}$$

$$z = \frac{k}{\beta^{|k-m|}}, 1 \leq \beta \leq 1.4 \tag{5b}$$

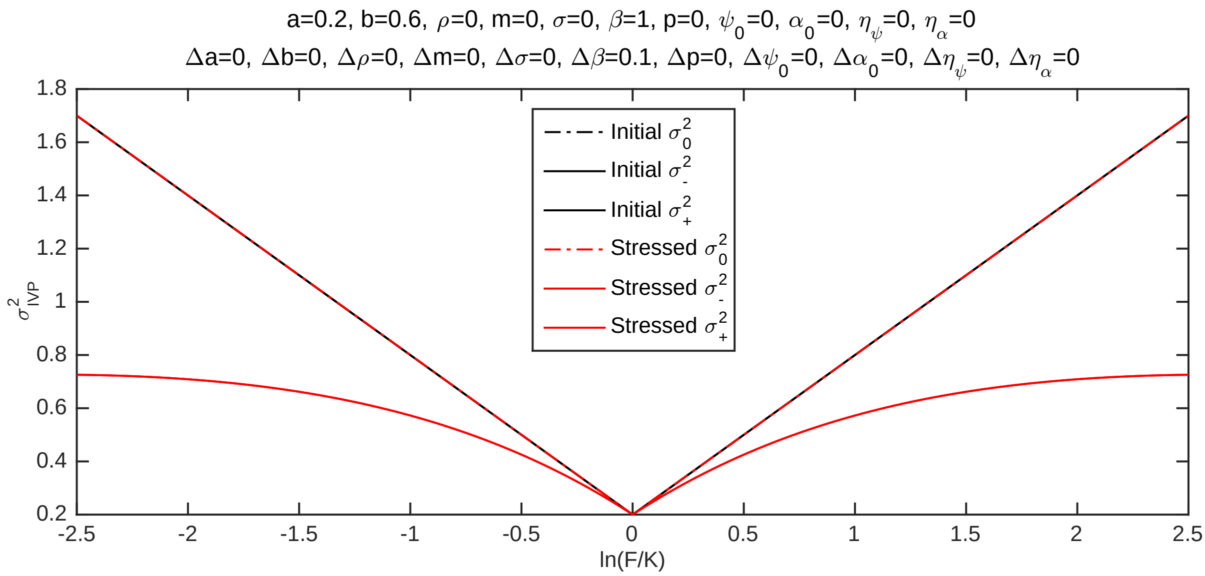


Figure 6: Impact of a change in β parameter in the gSVI/IVP model

Remark The downside transform in the gSVI [12] was arbitrarily given by $z = \frac{k}{\beta^{|k-m|}}, 1 \leq \beta \leq 1.4$. It is however, important to note, that there are many ways of defining the downside transform. One general approach would be to define μ and η like it is done in equation (6a). That idea can be prolonged to exp like function such as the one in equation (6b). The idea is always the same: the further away you are from the ATM, the bigger the necessary adjustment on the wings.

$$z = \frac{k}{\beta^{\mu+\eta|k-m|}} \tag{6a}$$

$$z = e^{-\beta|k-m|}(k-m) \tag{6b}$$

Mahdavi-Damghani, in introducing the IVP model [11] picked in equation (6a) a $\mu = 1$ and $\eta = 4$ and have the transformation in the form $z = \frac{k}{\beta^{1+4|k-m|}}$ because it yields better optimization results on the FX markets and also because it relaxes the constraint on β but our intuition is that the exp like function may work better when it comes to showing convergence between the modified Heston and the IVP model.

2.4.2 Market argument and data support

It is correct that the log-normal distribution has finite moment but it is widely accepted that the log-normal distribution models poorly the tails of the distributions as well. It would naturally be assumed that under these circumstances then the implied variance may grow slower than $|x|^{\frac{1}{2}}$ which corroborates with all the asset classes (Equities, Commodities, FX and Rates) which all exhibit implied variances which are sub-linear in log-moneyness (delta space for FX). Figure 7[5] exposes the superiority of the IVP' mid over the SVI model, more specifically the sub-linearity of the wings in variance space with market observable data.

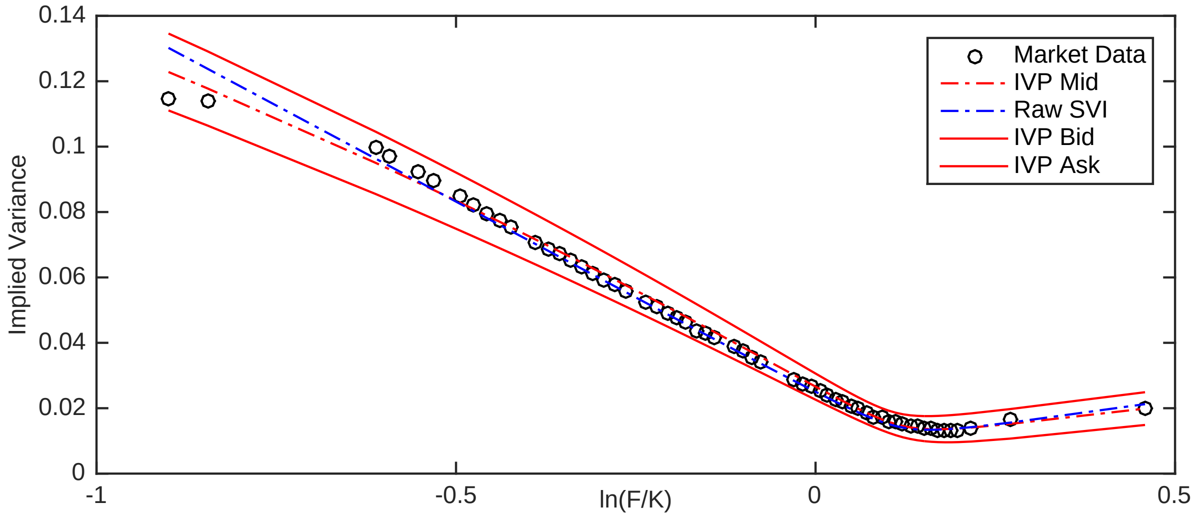


Figure 7: IVP vs SVI fit 1 year expiry S&P 500 index options [3] on 26/11/2016

3 The spot focused analysis

In this section we will guide the reader through the road which led us to understand the spot focused mathematical argument that makes the implied variance wings sub-linear from the traditional Heston set-up. First note, from figure 7 that the occasional sub-linearity of the wings is a market reality.

3.1 Heavy-tails, moment explosion & wings sub-linearity

When we first studied the spot focused mathematical argument we associated fat tails to moment explosions. This path proved misleading.

[5]Though this is not the main point of this paper, the IVP presents the added benefits of having a liquidity components which mathematical specification as been given by equation (45f) and which Bid-Ask has been incorporated in figure 7. The main point of the latter figure is however to expose the fact that the IVP's mid has a superior fit compared to the SVI model.

Proposition The log-normal distribution has finite moments of all orders despite being heavy-tailed.

Definition (Heavy-tailed distribution) The distribution of random variable X with cumulative distribution function (cdf) F is said to have heavy-tails if:

$$\lim_{x\to\infty} e^{kx} \Pr[X > x] = \infty \quad \text{for all } k > 0. \tag{7}$$

Lemma (Heavy-tails of log-normal distribution): The log-normal distribution has heavy-tails.

Proof If X is log-normal, then $Y = \log X$ is normal. Consider

$$\lim_{y\to\infty} e^{ky} \Pr[Y > y] = \lim_{y\to\infty} e^{ky} \frac{1}{\sqrt{2\pi}\sigma} e^{-(y-\mu)^2/(2\sigma^2)}\, dy. \tag{8}$$

Now observe that,

$$\begin{aligned} ky - \frac{(y-\mu)^2}{2\sigma^2} &= -\frac{-2k\sigma^2 y + y^2 - 2\mu y + \mu^2}{2\sigma^2} \\ &= -\frac{1}{2\sigma^2}\left(y^2 - 2(\mu + k\sigma^2)y + (\mu + k\sigma^2)^2 + \mu^2 - (\mu + k\sigma^2)^2\right) \\ &= -\frac{\left(y - (\mu + k\sigma^2)\right)^2}{2\sigma^2} + \frac{k(2\mu + k\sigma^2)}{2}. \end{aligned}$$

Consequently we have:

$$\lim_{y\to\infty} e^{ky} \frac{1}{\sqrt{2\pi}\sigma} e^{-(y-\mu)^2/(2\sigma^2)}\, dy = \lim_{y\to\infty} e^{k(2\mu+k\sigma^2)/2} \frac{1}{\sqrt{2\pi}\sigma} e^{-(y-\mu')^2/(2\sigma^2)}\, dy = 0.$$

since $Y = \log X$ then $\lim_{x\to\infty} e^{kx} \Pr[X > x] = \infty$.

Lemma (Moments of log-normal distribution): The log-normal distribution has finite moments of all orders.

Proof If X is log-normal, then $Y = \log X$ is normal. Consider

$$\mathrm{E}[X^k] = \mathrm{E}[e^{kY}] = \int_{y=-\infty}^{\infty} e^{ky} \frac{1}{\sqrt{2\pi}\sigma} e^{-(y-\mu)^2/(2\sigma^2)}\, dy. \tag{10}$$

Using the same mathematical techniques as in the last proof, the k^{th} moment is simply

$$\mathrm{E}[X^k] = e^{k(2\mu+k\sigma^2)/2} \int_{y=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-(y-\mu')^2/(2\sigma^2)}\, dy,$$

where $\mu' = \mu + k\sigma^2$. But this latter integral is equal to 1, being the integral of a normal density with mean μ' and variance σ^2. So $\mathrm{E}[X^k] = e^{k(2\mu+k\sigma^2)/2}$, therefore finite.

Gatheral and Jacquier explained that the SVI model is "consistent" with Roger Lees moment formula [9]. However Roger Lees exact claims are that "the moment formula has implications for skew extrapolation: it rejects functions that grow faster than $|x|^{\frac{1}{2}}$, and unless S_T has finite moments of all orders, it rejects those that grow slower than $|x|^{\frac{1}{2}}$". The higher moments of the log-normal distribution are actually finite even-though the distribution itself is considered heavy-tailed but not fat[6], and therefore mathematically the assumption on the linearity of the wings is rejected by that claim alone.

Remark There definitively lack of consensus over the use of the term heavy-tail. There are few of abuse of language that may create confusion for someone researching this topic for the first time. Few authors use the term to refer to distributions:

- which do not have all their power moments finite,
- whose moment generating function does not exist,
- which do not have a finite variance,
- which have heavier tails than the normal distribution.

The log-normal distribution is wrongly considered part of the fat tail distribution family due to the conception that the tails are fatter than the normal distribution but does not fulfill the formal definition of equation (7).

Remark The association heavy-tailed distribution inducing non finite higher moments is a common mistake. If you add the fact that the log-normal distribution is heavy but not fat, we can see how the problem could be confusing. We, ourselves had to review and rewrite the paper few times to make it clearer and added this and the last remark to help the reviewers see our point. It's possible Gatheral [5] made this common mistake? Gatheral and Jacquier [7] mention of Lee's statement on tail-wing behavior of implied variance was incompletethe claim became that the Black-Scholes implied variance was **exclusively** linear[7]. Later, Benaim and Fitz [14] improved Roger Lee's foment formula and concluded: "In models without moment explosion (Black-Scholes, [...]) the moment formula indicates sublinear behaviour of the implied variance".

3.2 The p-Heston argument

Assume the Heston model dynamic for the log-stock price is given by equation (11), split into a system of 3 equation where (11a) represents the dynamic of spot, (11b), the

[6]Note that the log-normal distribution is heavy-tailed but not fat-tailed, the latter being a distribution for which the probability density function, for large x, goes to zero as a power x^{-a}.

[7]It could have also been a deliberate negligence as the SVI was from inception presented as a parsimonious model and only selecting part of Roger Lee sentence was mathematically more convenient.

dynamics of vol and (11c) the relation between the correlation between spot and vol.

$$dS_t = \sqrt{v_t} S_t dW_t^1, \qquad S_0 \in \mathbb{R}_+^* \tag{11a}$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma v_t^p dW_t^2, \qquad v_0 \in \mathbb{R}_+^* \tag{11b}$$

$$d\left\langle W^1, W^2 \right\rangle_t = \rho dt, \tag{11c}$$

with $p = \frac{1}{2}$, $\rho \in [-1, 1]$, κ, $\theta, \sigma, v_0 > 0$, and the Feller condition satisfied when $2k\theta \geq \sigma^2$. This condition ensures that 0 is an unattainable boundary for the process $(v_t)_{t\geq 0}$.

Definition (p-Heston model): In equation set (11), we will define the relaxed p-Heston Model the modification from sub-equation (11b) in which p is no longer set to $\frac{1}{2}$ but in which, now, $0 < p < \frac{1}{2}$.

Remark Note that the p-Heston model still satisfies the main constraint of the CIR model which objective was to enforce positivity of the underlying SDE. However, it relaxes the p parameter which chosen value seems arbitrary set to $\frac{1}{2}$ for no apparent conceptual reasons[8]. Andersen & Piterbarg [1] suggested that the $p = \frac{1}{2}$ makes the stochastic process "tractable" which means that the pricing of a European option can be done in closed form[9] [8] without however much regard to data driven constraints such as goodness of fit in the wings away from the ATM.

Andersen & Piterbarg [1][10] define the class of stochastic volatility models for which the asset price (11a) does not have moment explosion. By modifying the Heston model (eg: relaxing the constraint on $p = \frac{1}{2}$ to $0 < p < \frac{1}{2}$), the higher moments no longer explode and we get our sub-linear wings.

Conjecture (p-Heston to IVP Convergence): Following the previous remark and results from this paper, we would like to propose that the asymptotic convergence from equating the relaxed p-Heston model where $0 < p < \frac{1}{2}$ gives for result the IVP model which downside transform change of parameter β behaves like equation (6a).

Proof The proof for this conjecture is left for the follow-up subsequent paper.

4 The correlation focused analysis

Another way to explain sub-linearity of the wings', instead of taking the traditional Heston model mixed with the moment generating function argument, is to take the assumption of a stochastic volatility model.

[8] For instance if positivity was the main concern why not set $p = 1$? Continuing the argument, if the concern was that as the underlying stochastic process gets close to 0 and that we are concern for a swift return to the long term mean why set p to $\frac{1}{2}$ and not something closer to 0?

[9] It's straightforward to calculate the expectation and the variance of the underlier this way.

[10] see Theorem 4.1.

4.1 Forewords on the Inferred Correlation and Market drivers

In order to present the alternative model, let us before recall the Inferred Correlation [10] model in order to incorporate it in the Heston model.

4.1.1 Rational

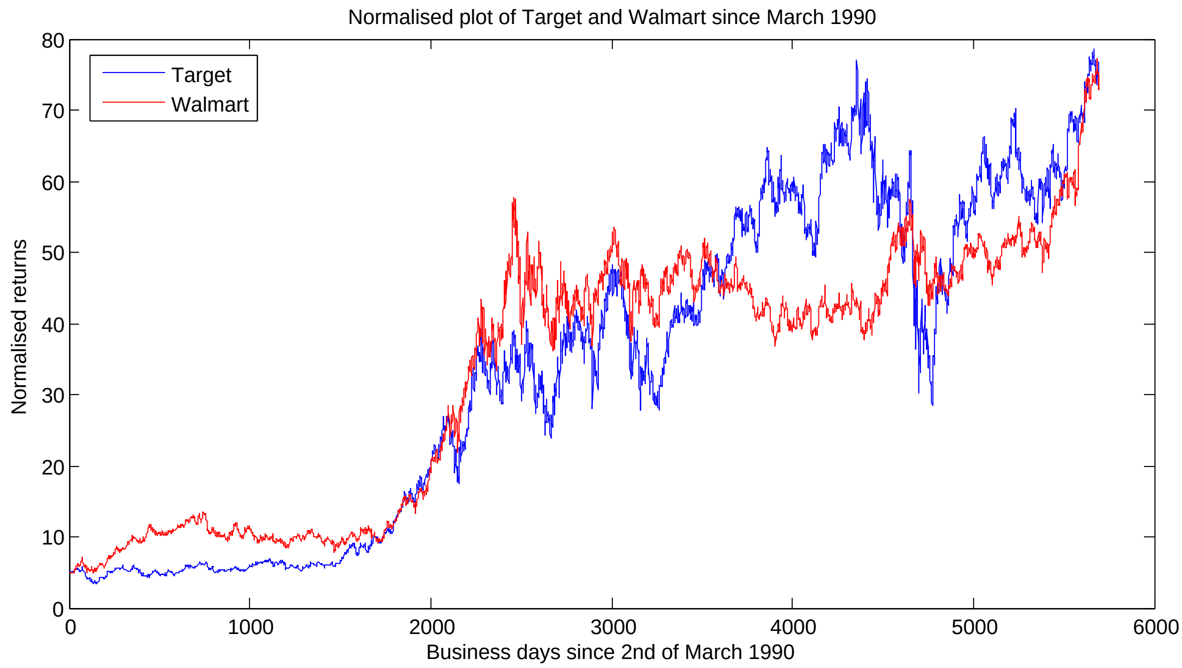
Mahdavi-Damghani [10] introduced the Inferred Correlation concepts by theorizing that the relationship of any two asset can be roughly decomposed of a short term and long term risk. The rational for the long term risk is that during the time of rare market crashes all assets tank. However, in the more bullish periods, the immediate risk takes the ascendant over the long term risk which visibility becomes less pronounced and the "macro" driver less visible. These influences are accompanied with small but still existent mean reversion forces from one asset to the other for which a substitution from one to the other exists. For instance within the commodities market, it cost a certain amount of money to turn WTI to Brent and therefore if one of these assets become relatively expensive, it becomes worth it to buy the other and pay for its transformation. Similarly, in other markets like equities, within one sector (eg: Target and Walmart from figure 8) competition within rivals may create similar mean reversion forces. Generally speaking if one product becomes expensive, it makes other products, sometimes, seemingly orthogonal, more attractive. The short term risk associated to the departure from this model is insured by both random events and high frequency index traders who hedge themselves on the market while structurers sell the relevant products.

4.1.2 Evidence

Traders understand that measured correlation does not accurately represent the relationship between assets in the long term as it fails to capture the long term drift of the underliers. As such traders understand that in situations where the perceived equilibrium between assets is away from it historical mean then the expectation of future realized correlation should be smaller than its historical mean correlation. A way to represent this concept mathematically is via the equation of a semi circle. Indeed if we call Ω the set of points (x, y) available for marketed correlation then the semi-circle that will best fit these market correlation can be retrieved using equation (12).

$$\{\hat{x}_c, \hat{y}_c, \hat{r}\} = \arg \min_{x_c, y_c, r} \sum_{i=1}^{N} [(x + x_c)^2 + (y + y_c)^2 - r^2], \ (x, y) \in \Omega \tag{12}$$

Note that figures 9 support the idea that traders believe that underliers have the same drift in the long run. Indeed these figures suggest that when the ratio of the underlier x over underlier y are away from some sort of natural mean then in anticipation that the



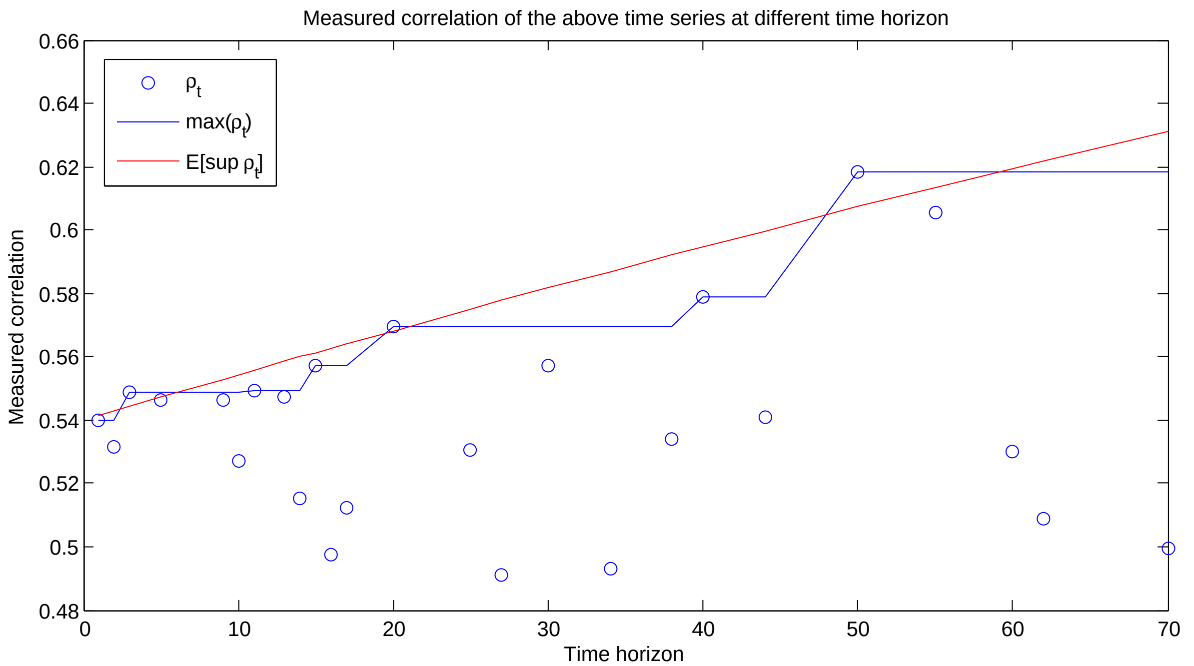


Figure 8: Time series of Target and Walmart in the last 22 years and statistics about their measured and inferred Correlation estimates

underliers should mean revert the traders mark an implied correlation smaller than if the ratio was at its mean.

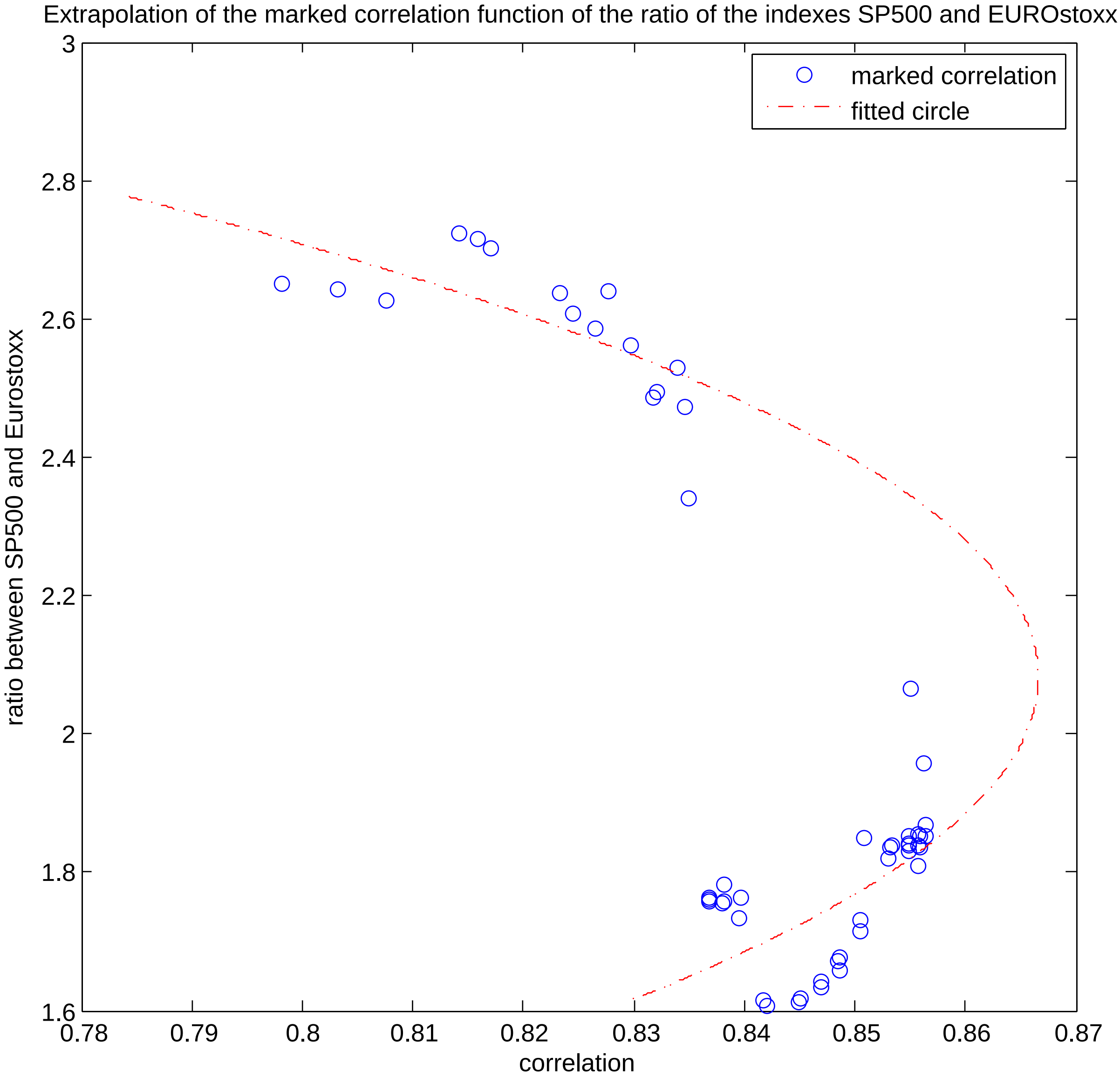


Figure 9: Marked correlation & the ratio of the relevant underliers fitted on a semi-circle.

4.2 Review & Formal Definition

4.2.1 Cointelation

Cointelation [10, 13] is a portmanteau neologism in finance, designed to signify a hybrid method between between the **cointe**gration and the corre**lation** models.

Remark: A caveat needs to be noted here: the term cointegration is differently formulated than the usual cointegration models presented in the econometrics literature. The term cointegration here represent a technical jargon introduced in the "misleading value of measured correlation" [13, 10] which aim is to specify the concept of mean reversion in the sense of the Ornstein-Uhlenbeck model [15].

Definition (Cointelation Model): Let $(\Omega, (f)_{(t\geq 0)}, \mathbb{P})$, be our probability space with

$(f)_{(t\geq 0)}$ generated by the $T+1$ dimensional Brownian motion and $\mathbb{P}$ the historical probability measure under which the discounted price of the underlier, rS, is not necessarily a martingale. The Cointelation model is defined by the set of 2 SDE's given by:

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t, \tag{13a}$$

$$dS_{l,t} = \theta(S_t - S_{l,t})dt + \sigma_l S_{l,t} dW_t^l, \tag{13b}$$

$$d\langle W_t, W_t^l \rangle = \rho dt, \tag{13c}$$

where ρ is alternatively called the Correlation of the Cointelation, instantaneous Correlation or the infinitesimal Correlation. The process $(S)_{t\geq 0}$ is called the leading process, $(S)_{l,t\geq 0}$ the lagging process, r the drift, the σ the volatility and θ the speed of mean reversion.

We note that setting $\theta = 0$ in sub-equation (13b) yields the classic correlation model. Conversely, setting $\rho = 0$ in sub-equation (13c) yields our version of the cointegration model (essentially an Ornstein-Uhlenbeck process [15] with a stochastic mean and an $S_{l,t}$ in front of the stochastic part of the SDE to enforce positivety).

Proposition Using Pearson's correlation measure[11], one notable interesting property of the Cointelation model is that its inferred correlation mirror may hit the whole measured correlation spectrum $[-1, 1]$ depending on the choice of $\rho = -1$ and θ.

4.2.2 Inferred Correlation

The proposed Cointelation test [10] is subdivided in 4 steps including the inferred correlation conjecture. The idea being that if one takes the discrete version of equation (13a), the correlation that you measure as a function of the timescale would increase faster as θ increases.

Conjecture (Inferred Correlation): We consider a stock price process $(S_t)_{t\geq 0}$ with natural filtration $(\mathcal{F}_t)_{t\geq 0}$, and we define the forward price process $(F_t)_{t\geq 0}$ by $F_t :=$ $\mathbb{E}(S_t|\mathcal{F}_0)$ then considering, the dynamics of equation (13), we have:

$$\rho_\tau^* \approx \rho + (1-\rho)\left[1 - \exp\left(-\theta\lambda(\tau - 1)\right)\right] \tag{14}$$

where $\rho_\tau^* = \mathbb{E}[\sup_{0<t\leq\tau} \rho_t]$, $\tau \in \mathbb{Z}^*, \theta \in [0, 1]$

Remark: The author [10] has set $\lambda \approx 1.75$ *for "regular financial data"*. The author explains that λ is actually itself a function of the other parameters. Though this approximation provides interesting empirical results, this latter point is an open problem with the Cointelation test as of now.

[11] Therefore independent from the classic ways of tempering in with measuring correlation

4.3 Heston model with a local correlation

Both Heston and SVI are very popular in the industry and converge asymptotically to each other:

$$dS_t = \sqrt{v_t} S_t dW_t^1, \quad S_0 \in \mathbb{R}_+^* \tag{15a}$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma v_t^{\frac{1}{2}} dW_t^2, \quad v_0 \in \mathbb{R}_+^* \tag{15b}$$

$$d\left\langle W^1, W^2 \right\rangle_t = \rho dt, \tag{15c}$$

$$v(k,t) \to a + b[\rho(k-m) + \sqrt{(k-m)^2 + \sigma^2}] \tag{15d}$$

The SVI was made obsolete at Bank of America for its inability to accurately model the sub-linearity of the wings. At the same time the Heston model is known for its inability to model the smile. This convergence is mathematically convenient for the SVI but does not solve an economic and data driven fact (see Figure 7). Only an implied correlation surface could reconcile the current models to the data. The proposed stochastic volatility model that we speculate as being able to converge towards the IVP model if given by equation (16).

$$dS_t = \sqrt{v_t} S_t dW_t^1, \quad S_0 \in \mathbb{R}_+^* \tag{16a}$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_t^2, \quad v_0 \in \mathbb{R}_+^* \tag{16b}$$

$$d\left\langle W^1, W^2 \right\rangle_t = \rho(t, S_t) dt, \tag{16c}$$

$$\rho(t, S_t) = \rho_+(t) + [\rho_-(t) - \rho_+(t)]\,[1 - \exp(-\beta(t)|S_t - K|)] \tag{16d}$$

$$v(k,t) \to a + b[\rho(z-m) + \sqrt{(z-m)^2 + \sigma^2}] \tag{16e}$$

$$z = h(k) \tag{16f}$$

with $h(k)$ an increase sub-linear function[12]. The main differences as compared to the classic Heston equation model is the ρ becoming stochastic and given by $\rho(t, S_t)$. The IVP is the best parametrization of the volatility surface for modeling the wings and therefore reconciling stochastic and implied volatility. The proof has not been written yet and the IVP's β parameter (downside transform) is more likely to be an exp function rather than a power function.

Conjecture (Inferred local correlation to IVP Convergence): Following the previous points and results from this paper, we would like to propose that the asymptotic convergence from equating the Heston model with Inferred local Correlation gives for result the IVP model which downside transform change of parameter β behaves like equation (6a).

[12]for example $z = \frac{k}{\beta^{|k-m|}}, 1 \leq \beta \leq 1.4$ like we have seen earlier [12, 11].

5 Conclusion

5.1 Summary

We have given a mathematical and a market argument on the sub-linearity of the wings for the implied variance. We have exposed that Gatheral's parsimonuous stochastic volatility inspired (SVI) parameterization claim to have two key properties that have led to its subsequent popularity with practitioners has a critically limiting factor in the wings which he presents as linear but which market observable data shows as potentially sometimes sub-linear. Though some quality work has been done in that domain [14, 5, 2, 9], the business application of the SVI was decommissioned in the industry. We explain this apparent contradiction by pointing to an mathematically convenient[13] factor in the Heston model which we expose and consequently introduce the p-Heston model instead. The link between the latter and the SVI being broken, we propose a connection to the Implied Volatility surface Parametrisation (IVP) [11, 12] recently introduced and proposed a conjecture between its mirror convergence with this p-Heston model using the parallel between the recently published Heston to SVI convergence [5].

5.2 Puzzle of the proof in 3 steps

The way we have organized this paper is really the way we anticipate the proof for the convergence of the IVP model to the p-Heston model to take place. First, similarly to section 2.4 where we wrote equation (44b) in the following form:

$$\sigma^2_{\mathrm{IVP}}(x) = \frac{w_1}{2}\left[1 + w_2\rho z + \sqrt{(w_2 z+\rho)^2 + (1-\rho^2)}\right], \tag{17}$$

with $z = \frac{x}{\beta^{1+4|x-m|}}$, where x is time-scaled logmoneyness. We would need to choose the w_1 and w_2 parameters in terms of the p-Heston model as in [5] so slightly differently to the below[14]:

$$w_1 := \frac{4\kappa\theta}{\sigma^2(1-\rho^2)}\left(\sqrt{(2\kappa-\rho\sigma)^2 + \sigma^2(1-\rho^2)} - (2\kappa-\rho\sigma)\right), \quad \text{and} \quad w_2 := \frac{\sigma}{\kappa\theta}. \tag{18}$$

Finally we would need to assess the following proposition:

Proposition Under the Assumption $k-\rho\sigma > 0$ and choice of parameters (18) $\sigma^2_{\mathrm{IVP}}(x) = \sigma^2_{\infty}(x) + \varepsilon(x,t)$ for all $x \in \mathbb{R}$, with x being time-scaled log-moneyness.

Claim (Heston to IVP): By a similar process: from lemma 2.3 we know that we can potentially transform equation 44b to the form in the equation (17), lemma A.2 shows

[13] As opposed to *mathematically exact* in terms of modeling the reality of the market.

[14] We may want to take a look at some of Forde's work [4].

the convergence of Heston to Natural SVI without the error term and Conjecture (Error Downside Transform) which we anticipate would do the trick.

Remark One important point to take away from this paper is the fact that using convenient mathematics (eg: $p = \frac{1}{2}$) as opposed to mathematics that genuinely models the financial world (eg: sub-linearity of the wings) may create a branch of challenging mathematics that may however become obsolete quickly.

A Convergence of Heston to natural SVI

A.1 Heston implied variance

Forde et al. proved[15] that the implied volatility surface [4] behaves asymptotically like equation (19).

$$\sigma_\infty^2(x) = \begin{cases} 2\left(2V^*(x) - x + \sqrt{V^*(x)^2 - kV^*(x)}\right), & \text{for } x \in \mathbb{R}\setminus\left[-\frac{1}{\theta}, \frac{1}{\hat{\theta}}\right], \\ 2\left(2V^*(x) - x - \sqrt{V^*(x)^2 - kV^*(x)}\right), & \text{for } x \in \left(-\frac{1}{\theta}, \frac{1}{\hat{\theta}}\right). \end{cases} \tag{19}$$

with $\hat{\theta} := \kappa\theta/(\kappa - \rho\sigma)$, and the function $V^* : \mathbb{R} \mapsto \mathbb{R}_+$ is defined by

$$V^*(x) := p^*(x)x - V(p^*(x)), \quad \text{for all } x \in \mathbb{R}, \tag{20}$$

where

$$V(p) := \frac{\kappa\theta}{\sigma^2}\left(\kappa - \rho\sigma p - d(p)\right), \quad \text{for all } p \in (p_-, p_+), \tag{21a}$$

$$d(p) := \sqrt{(\kappa - \rho\sigma p)^2 + \sigma^2 p(1 - p^2)}, \quad \text{for all } p \in (p_-, p_+), \tag{21b}$$

$$p^*(x) := \frac{\sigma - 2\kappa\rho + (\kappa\theta\rho + x\sigma)\eta(x^2\sigma^2 + 2x\kappa\theta\rho\sigma + \kappa^2\theta^2)^{-\frac{1}{2}}}{2\sigma\hat{\rho}^2}, \quad \text{for all } x \in \mathbb{R}, \tag{21c}$$

$$\eta := \sqrt{4k^2 + \sigma^2 - 4\kappa\rho\sigma}, \tag{21d}$$

$$p_\pm := \left(-2\kappa\rho + \sigma \pm \sqrt{\sigma^2 + 4\kappa^2 - 4\kappa\rho\sigma}\right)/(2\sigma\hat{\rho}^2), \tag{21e}$$

$$\hat{\rho} := \sqrt{1 - \rho^2} \tag{21f}$$

Remark Note that here the implied variance corresponds to European options with maturity T and maturity-dependent strike $K = S_0 \exp(xT)$.

A.2 Heston to natural SVI

The material associated with this part of the proof has been taken from [7].

Lemma (Heston to natural SVI): The natural SVI parametrization for implied variance is given by equation[16] (22)

$$\sigma_{\text{SVI}}^2(x) = \frac{w_1}{2}\left(1 + w_2\rho x + \sqrt{(w_2 x + \rho)^2 + (1 - \rho^2)}\right), \quad \forall x \in \mathbb{R}, \tag{22}$$

where x corresponds to a time-scaled log-moneyness. With the choice of SVI parameters

[15] We refer here to Forde's original paper [4] for this long and convoluted proof. We will assume in that paper that the proof is correct.

[16] Note that here the Δ parameter in equation (2), which appears in natural SVI formula in [7] is set to be equal 0 in equation (22).

in terms of the Heston parameters are the following:

$$w_1 := \frac{4\kappa\theta}{\sigma^2(1-\rho^2)}\left(\sqrt{(2\kappa-\rho\sigma)^2+\sigma^2(1-\rho^2)}-(2\kappa-\rho\sigma)\right), \quad \text{and} \quad w_2 := \frac{\sigma}{\kappa\theta}. \tag{23}$$

and under Assumption $k-\rho\sigma>0$ (23), $\sigma^2_{SVI}(x)=\sigma^2_\infty(x)$ for all $x\in\mathbb{R}$.

Proof Denote $\Delta(x) := \sqrt{\sigma^2x^2+2\kappa\theta\rho\sigma x+\kappa^2\theta^2}$, where $\eta=\sqrt{4k^2+\sigma^2-4\kappa\rho\sigma}$ and $\hat\rho$ is defined by (21f). By plugging (23) into (22) the SVI implied variance formula is of the following form

$$\sigma^2_{SVI}(x)=\frac{2}{\sigma^2\hat\rho^2}\left(\eta-(2\kappa-\rho\sigma)\right)\left(\kappa\theta+\rho\sigma x+\Delta(x)\right), \quad \text{for all} x\in\mathbb{R}. \tag{24}$$

In order to simplify the expression for σ^2_∞ in (19), first, the expression for $V^*(x)$ in 19 is rewritten in the following form:

$$V^*(x)=\frac{A(x)\Delta(x)+B(x)\eta}{2\sigma^2\hat\rho^2\Delta(x)} \tag{25}$$

with

$$A(x) := x\sigma^2-2x\kappa\rho\sigma-2\kappa^2\theta+\kappa\theta\rho\sigma, \quad \text{and} \quad B(x) := 2x\sigma\kappa\theta\rho+x^2\sigma^2+\kappa^2\theta^2\rho^2+\kappa^2\theta^2\hat\rho^2. \tag{26}$$

Since $B(x)=\Delta^2(x) \implies V^*(x)=(A(x)+\Delta(x)\eta)/(2\sigma^2\hat\rho^2)$. And

$$2V^*(x)-x=\frac{A(x)+\Delta(x)\eta-x\sigma^2\hat\rho^2}{\sigma^2\hat\rho^2}=\frac{\Delta(x)\eta-(2\kappa-\rho\sigma)(\kappa\theta+x\rho\sigma)}{\sigma^2\hat\rho^2}, \tag{27}$$

where the factorization $A(x)-x\sigma^2\hat\rho^2=-(2\kappa-\rho\sigma)(\kappa\theta+x\rho\sigma)$ is used.
Now in (19) denote $\Phi(x) := V^*(x)^2-xV^*(x)$. We have

$$\Phi(x)=\left(\frac{\Delta(x)\eta}{2\sigma^2\hat\rho^2}\right)^2+\alpha(x)\Delta(x)+\beta(x), \tag{28}$$

where

$$\alpha(x) := -\frac{\eta(2\kappa-\rho\sigma)(\kappa\theta+x\rho\sigma)}{2\sigma^4\hat\rho^4}, \quad \text{and} \quad \beta(x) := \frac{1}{4\sigma^4\hat\rho^4}\left((2\kappa-\rho\sigma)^2(\kappa\theta+x\rho\sigma)^2-x^2\sigma^4\hat\rho^4\right).$$

Using the following factorization:

$$\Delta^2(x)=(\kappa\theta+x\rho\sigma)^2+x^2\sigma^2\hat\rho^2, \quad \text{and} \quad \eta^2=(2\kappa-\rho\sigma)^2+\sigma^2\hat\rho^2 \tag{29}$$

we can write $\beta(x) = (4\sigma^4\hat{\rho}^4)^{-1}\left((2\kappa-\rho\sigma)^2\Delta^2(x) - x^2\sigma^2\hat{\rho}^2\eta^2\right)$ and hence

$$\begin{aligned}
\Phi(x) &= \frac{1}{4\sigma^4\hat{\rho}^4}\left(\left[(2\kappa-\rho\sigma)^2+\sigma^2\hat{\rho}^2\right]\Delta^2(x)+\alpha(x)\Delta(x)+(\eta^2-\sigma^2\hat{\rho}^2)(\Delta^2(x)-x^2\sigma^2\hat{\rho}^2)-x^2\sigma^4\hat{\rho}^4\right)\\
&= \frac{1}{4\sigma^4\hat{\rho}^4}\left((2\kappa-\rho\sigma)^2\Delta^2(x)+\alpha(x)\Delta(x)+\eta^2(\kappa\theta+x\rho\sigma)^2\right)\\
&= \frac{1}{4\sigma^4\hat{\rho}^4}\left(\eta(\kappa\theta+x\rho\sigma)-(2\kappa-\rho\sigma)\Delta(x)\right)^2, \qquad (30)
\end{aligned}$$

where $a(x) \coloneqq 4\sigma^4\hat{\rho}^4\alpha(x)$. By taking the square root of $Phi(x)$ we have

$$\begin{aligned}
&\eta(\kappa\theta+x\rho\sigma)-(2\kappa-\rho\sigma)\Delta(x)\\
&= (\kappa\theta+x\rho\sigma)\sqrt{(2\kappa-\rho\sigma)^2+\sigma^2\hat{\rho}^2}-(2\kappa-\rho\sigma)\sqrt{(\kappa\theta+x\rho\sigma)^2+x^2\sigma^2\hat{\rho}^2}\\
&= \sqrt{\gamma(x)+\sigma^2\hat{\rho}^2(\kappa\theta+x\rho\sigma)^2}-\sqrt{\gamma(x)+x^2\sigma^2\hat{\rho}^2(2\kappa-\rho\sigma)^2},
\end{aligned}$$

where $\gamma(x) \coloneqq (2\kappa-\rho\sigma)^2(\kappa\theta+x\rho\sigma)^2$. Now, because $\gamma(x) \geq 0$ for all $x \in \mathbb{R}$, then the sign of this whole expression is simply given by the difference $\psi(x) \coloneqq \sigma^2\hat{\rho}^2(\kappa\theta+x\rho\sigma)^2 - x^2\sigma^2\hat{\rho}^2(2\kappa-\rho\sigma)^2$. Note further that we actually have $\psi(x) = \kappa\sigma^2\hat{\rho}^2(2x+\theta)(2x\rho\sigma+\kappa\theta-2\kappa x)$, that this polynomial has exactly two real roots $-\theta/2$ and $\hat{\theta}/2$, and that its second-order coefficients reads $-4\kappa\sigma^2\hat{\rho}^2(\kappa-\rho\sigma) < 0$ under assumption $\kappa-\rho\sigma > 0$. So plugging (27) and (30) into (19), we exactly obtain (24) and the proposition follows.

Remark In their original paper [4], Forde, Jaquier and Mijatovic show that

$$\hat{\sigma}_t^2(x) = \hat{\sigma}_\infty^2(x) + t^{-1}\frac{8\sigma_\infty^4(x)}{4x^2-\hat{\sigma}_\infty(x)}\log\left(\frac{A(x)}{A_{BS}(x,\hat{\sigma}_\infty(x),0)}\right)+o(t^{-1}) \qquad (31)$$

with $A(x)$ a complicated function which detailed specification can be found in the original paper [4]. Using Forde, Jaquier and Mijatovic [4] notation $\hat{\sigma}_t^2(x) = \hat{\sigma}_\infty^2(x)+\varepsilon(t,x)$, where $\varepsilon(t,x)$ represents the error term $t^{-1}\frac{8\sigma_\infty^4(x)}{4x^2-\hat{\sigma}_\infty(x)}\times\log\left(\frac{A(x)}{A_{BS}(x,\hat{\sigma}_\infty(x),0)}\right)+o(t^{-1})$ then it is tempting to assume:

1. $\varepsilon(t,x)$ behaves like the downside transform family introduced by equation (6a).
2. the 2 in $\hat{\sigma}_t^2(x)$ can be replaced by $4p$ with $0 < p < \frac{1}{2}$ and assume the sub-linearity would magically come out through a tedious derivation.

However none of these 2 tempting guess would work because the moments are now all finite and there is no finite interval of study needed and the SVI and IVP equations are slightly different. You have to derive the problem from scratch.

B Proof for right and left wing sub-linearity

The following two theorems are taken from [14].
We consider risk-neutral returns X with cummulative distribution function F, wit $\bar{F} = 1 - F$ and f for the probability density function of X. The class of regularly varying functions at $+\infty$ of index α is denoted by $\mathbb{R}_\alpha$.

Theorem (Right-Tail-Wing Formula) Assume $\alpha > 0$ and

$$\exists \epsilon > 0 : \quad \mathbb{E}\left[e^{(1+\epsilon)X}\right] < \infty. \tag{32}$$

Then (i) $\to$ (ii) $\to$ (iii) $\to$ (iv),
where

$$-\log f(k) \in \mathbb{R}_\alpha \tag{33a}$$

$$-\log \bar{F}(k) \in \mathbb{R}_\alpha \tag{33b}$$

$$-\log c(k) \in \mathbb{R}\alpha \tag{33c}$$

and[17]

$$V(k)^2/k \sim \psi\left[-1 - \log \bar{F}(k)/k\right]. \tag{34}$$

If (33b) holds then $-\log c(k) \sim -k - \log \bar{F}$ and

$$V(k)^2/k \sim \psi\left[-1 - \log \bar{F}(k)/k\right], \tag{35}$$

if (33a) holds, then $-\log f \sim -\log \bar{F}$ and

$$V^2(k)/k \sim \psi\left[-1 - \log f(k)/k\right]. \tag{36}$$

Finally, if either $-\log f(k)/k$ or $-\log \bar{F}(k)/k$ or $-\log c(k)/k$ goes to infinity as $k \to \infty$ then $V^2(k)$ behaves sublinearly. More precisely,

$$V^2(k)/k \sim \frac{1}{-2\log f(k)/k} \quad \text{or} \quad \frac{1}{-2\log \bar{F}(k)/k} \tag{37}$$

Theorem (Left-Tail-Wing Formula) Assume $\alpha > 0$ and

$$\exists \epsilon > 0 : \quad \mathbb{E}\left[e^{-\epsilon X}\right] < \infty. \tag{38}$$

Then (i) $\to$ (ii) $\to$ (iii) $\to$ (iv),

[17] $g(k) \sim h(k)$ means $g(k)/h(k) \to 1$ as $k \to \infty$

where

$$-\log f(-k) \in \mathbb{R}_\alpha \tag{39a}$$

$$-\log F(-k) \in \mathbb{R}_\alpha \tag{39b}$$

$$-\log p(-k) \in \mathbb{R}\alpha \tag{39c}$$

and

$$V(-k)^2/k \sim \psi\left[-1 - \log p(-k)/k\right]. \tag{40}$$

If (39b) holds then $-\log p(k) \sim k - \log F(-k)$ and

$$V(-k)^2/k \sim \psi\left[-\log F(-k)/k\right], \tag{41}$$

if (39a) holds, then the $-logf(-k) \sim -\log F(-k)$ and

$$V(k)^2/k \sim \psi\left[-\log f(-k)/k\right]. \tag{42}$$

Finally, if either $-\log f(-k)/k$ or $-\log F(-k)/k$ or $-\log p(-k)/k$ goes to infinity as $k \to \infty$ then $V^2(-k)$ behaves sublinearly. More precisely,

$$V^2(-k)/k \sim \frac{1}{-2\log f(-k)/k} \quad \text{or} \quad \frac{1}{-2\log F(-k)/k} \quad or \quad \frac{1}{-2\log p(-k)/k}. \tag{43}$$

C More about the IVP model

The IVP has other benefits than the downside transform such as the ability to model liquidity but these are more or less outside the scope of this specific paper. We will however summarize the parameters so the reader can focus on the part that are important.

The Implied Volatility surface Parametrization (IVP) equation

By incorporating the information on the gSVI, the ATM Bid Ask spread and the curvature adjustment of the wings Mahdavi-Damghani [12, 11] defines what he labeled the Implied Volatility surface Parametrization (IVP) split with its mid in equation (44b) and its liquidity parameters in equation (45f).

$$\sigma^2_{IVP,o,\tau}(k) = a_\tau + b_\tau\left[\rho_\tau\left(z_{o,\tau} - m_\tau\right) + \sqrt{\left(z_{o,\tau} - m_\tau\right)^2 + \sigma_\tau^2}\right] \tag{44a}$$

$$z_{o,\tau} = \frac{k}{\beta_{o,\tau}^{1+4|k-m|}} \tag{44b}$$

$$\sigma^2_{IVP,+,\tau}(k) = a_\tau + b_\tau\left[\rho_\tau\left(z_{+,\tau} - m_\tau\right) + \sqrt{\left(z_{+,\tau} - m_\tau\right)^2 + \sigma_\tau^2}\right] + \alpha_\tau(p) \tag{45a}$$

$$z_{+,\tau} = z_{o,\tau}[1 + \psi_\tau(p)] \tag{45b}$$

$$\sigma^2_{IVP,-,\tau}(k) = a_\tau + b_\tau\left[\rho_\tau\left(z_{-,\tau} - m_\tau\right) + \sqrt{\left(z_{-,\tau} - m_\tau\right)^2 + \sigma_\tau^2}\right] - \alpha_\tau(p) \tag{45c}$$

$$z_{-,\tau} = z_{o,\tau}[1 - \psi_\tau(p)] \tag{45d}$$

$$\alpha_\tau(p) = \alpha_{0,\tau} + (a_\tau - \alpha_{0,\tau})(1 - e^{-\eta_{\alpha_\tau} p}) \tag{45e}$$

$$\psi_\tau(p) = \psi_{0,\tau} + (1 - \psi_{0,\tau})(1 - e^{-\eta_{\psi_\tau}} p) \tag{45f}$$

Remark Note that once Bid Ask has been incorporated, we care a bit less about the mid in the context of vanilla options market making. Though the mid may have arbitrages at the portfolio level, the Bid-Ask relaxes the butterfly (equation (46a)) and calendar (equation (46b)) spread equations.

$$\forall\Delta, C(K - \Delta, \sigma_{IVP,+,t}(k)) - 2C(K, \sigma_{IVP,-,t}(k)) + C(K + \Delta, \sigma_{IVP,+,t}(k)) > 0 \tag{46a}$$

$$C(K, T + \Delta, \sigma_{IVP,+,t}(k)) - C(Ke^{-r\Delta}, T, \sigma_{IVP,-,t}(k)) \geq 0 \tag{46b}$$

The other parameters

The functions $\alpha(p)$ (figure 10) and ψp (figure 11) model the ATM and wing curvature of the Bid-Ask keeping in mind the idea that the bigger the position size the bigger the market impact and hence the wider the Bid-Ask. This market impact parameter is controlled by p (figure 12). Finally, couple of additional parameters model the elasticity of the liquidity: η_ψ (figure 13) and η_α (figure 14).

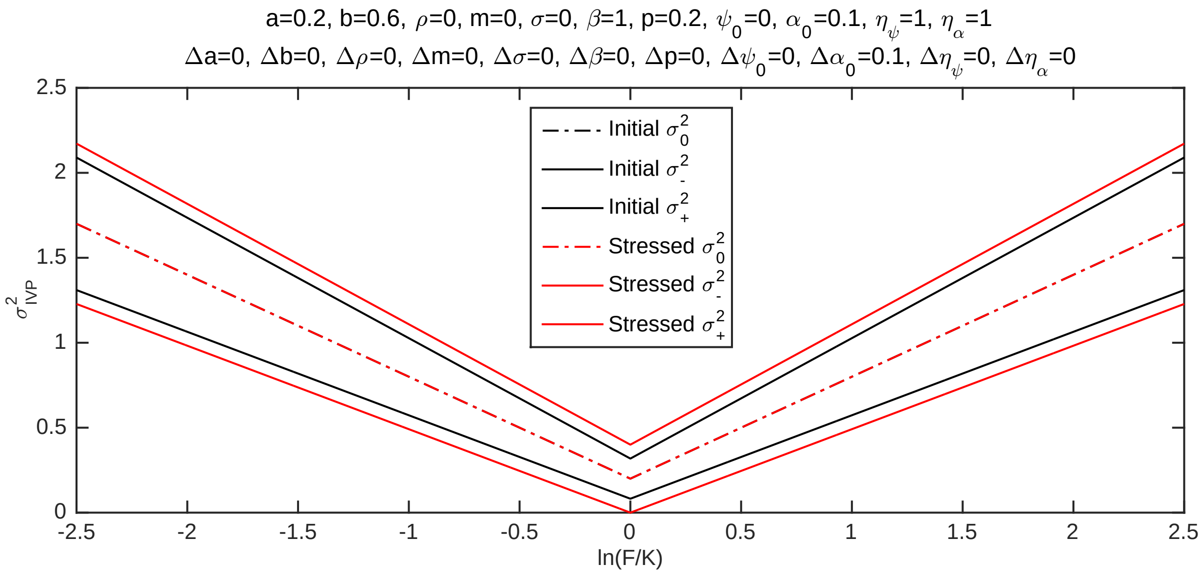


Figure 10: Impact of a change in the α parameter in the IVP model

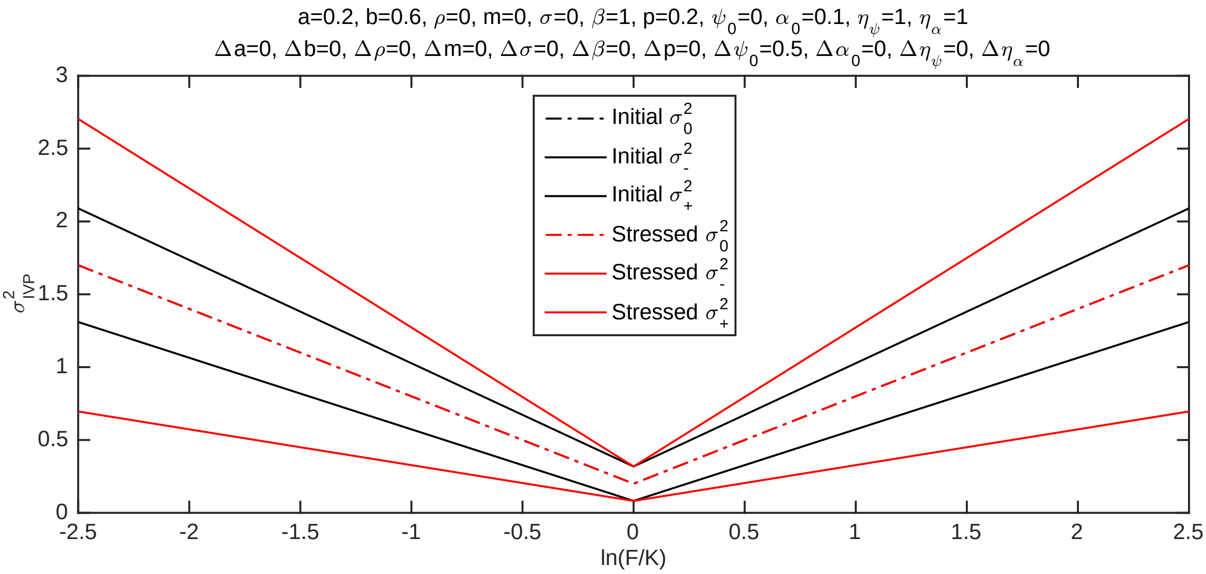


Figure 11: Impact of a change in the ψ parameter in the IVP model

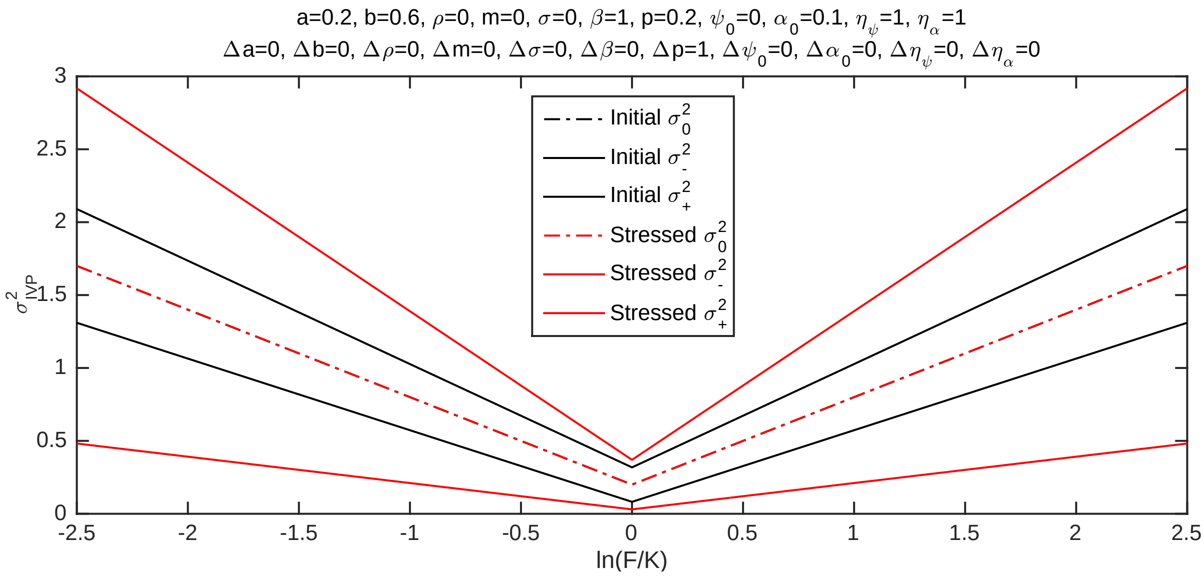


Figure 12: Impact of a change in the p parameter in the IVP model

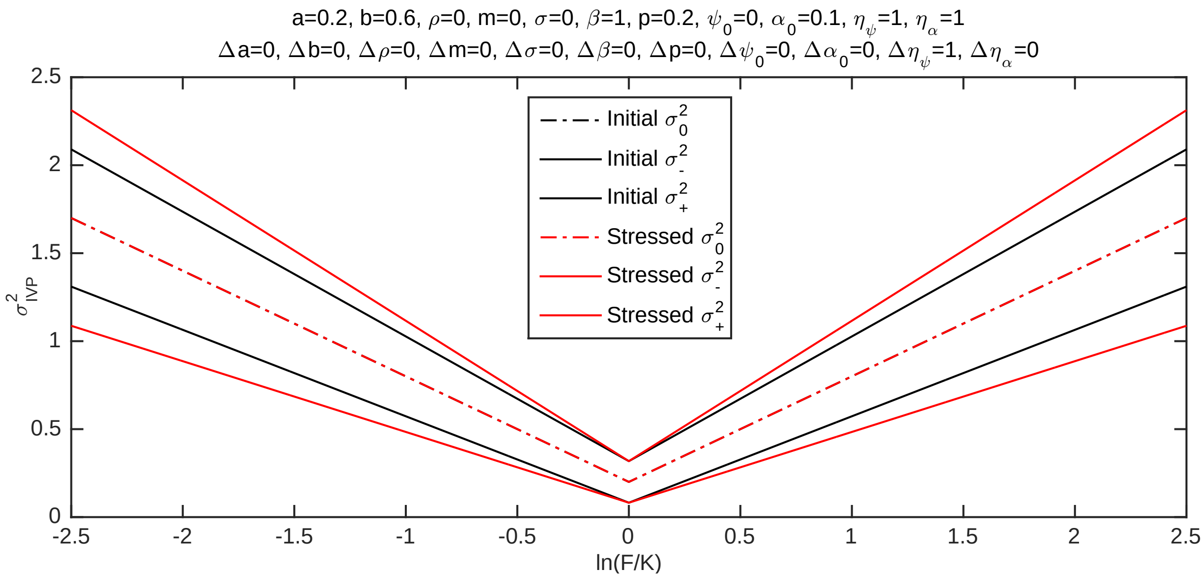


Figure 13: Impact of a change in the η_ψ parameter in the IVP model

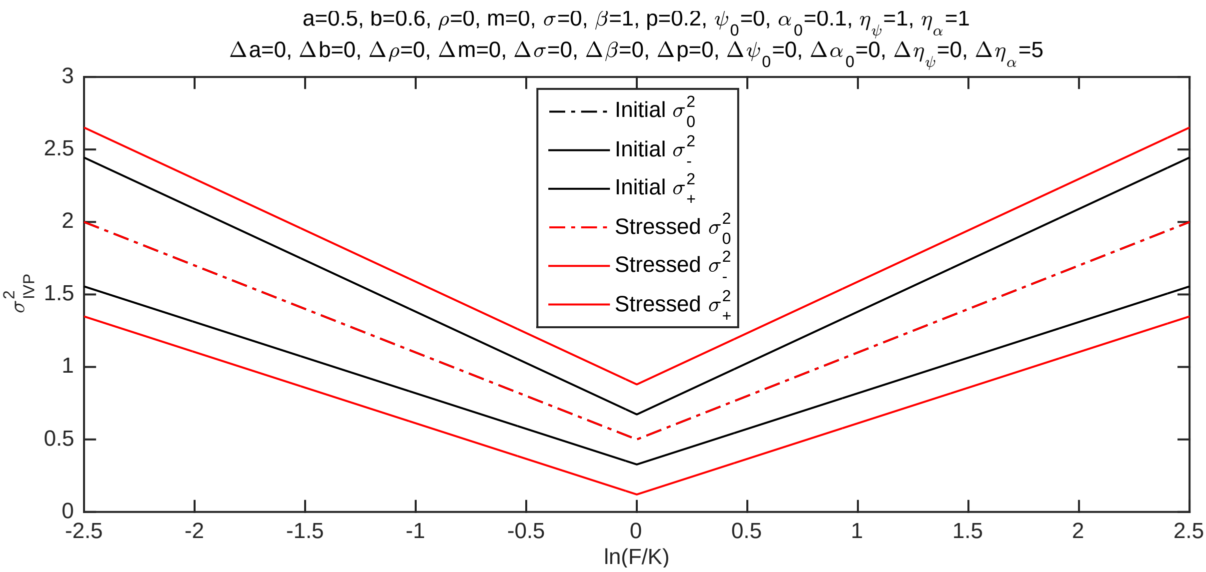


Figure 14: Impact of a change in the η_α parameter in the IVP model

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